

Sur le problème de Goursat concernant les équations différentielles hyperboliques du second ordre

par

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Présenté par T. WAŻEWSKI le 18 Mars, 1957

Considérons le problème de Goursat qui concerne la question de l'existence d'une solution de l'équation

$$(1) \quad z_{xy}(x, y) = f(x, y, z(x, y), z_x(x, y), z_y(x, y)),$$

admettant les valeurs données le long des deux courbes ayant un seul point d'intersection.

Nous allons donner dans la présente note une nouvelle démonstration de l'existence locale de la solution unique du problème énoncé ci-dessus — théorème 1 — en admettant pour la fonction $f(x, y, z, p, q)$ l'hypothèse de la continuité et celle de Lipschitz par rapport aux variables z, p, q . Un raisonnement simple *) permet de déduire du théorème 1 le théorème énoncé à la p. 77 de [3]. Le prolongement de la solution, définie dans un voisinage du point d'intersection de courbes données, appliqué par G. Stampacchia, permet à son tour de déduire du théorème 1, le théorème assurant l'existence globale de la solution unique du problème de Goursat (cf. [3] p. 81).

En renonçant à l'hypothèse que la fonction $f(x, y, z, p, q)$ satisfait à la condition de Lipschitz par rapport à la variable z , nous démontrons l'existence locale d'une solution (pas forcément unique) du problème de Goursat **). Cette démonstration est basée sur le théorème de Schauder sur le point fixe ([2], Satz 1).

*) Cf. [3], p. 78. Le théorème I, sans être énoncé explicitement, est aussi démontré dans [3] (§ I et § II, 1). Notre démonstration nous semble plus simple.

**) Il nous semble que dans tous les théorèmes publiés jusqu'ici concernant l'existence d'une solution du problème de Goursat, on a admis l'hypothèse que la fonction $f(x, y, z, p, q)$ satisfait à la condition de Lipschitz par rapport à la variable z . Dans le cas du problème de Darboux cette hypothèse fut supprimée par P. Hartman et A. Wintner (cf. [1], (I), p. 836).

1. Désignons par R le rectangle défini par les inégalités

$$-a \leq x \leq a \quad \text{et} \quad -\beta \leq y \leq \beta, \quad \text{où} \quad 0 < a < \infty \quad \text{et} \quad 0 < \beta < \infty,$$

par k le nombre

$$(2) \quad k = 2 \max(a, \beta),$$

et par Π l'ensemble

$$|x| \leq a, \quad |y| \leq \beta, \quad |z|, |p|, |q| < \infty.$$

Considérons deux courbes Γ et Λ situées dans le rectangle R et définies par les équations respectives

$$y = \gamma(x), \quad \text{lorsque} \quad -a \leq x \leq a, \quad \gamma(0) = 0,$$

$$x = \lambda(y), \quad \text{lorsque} \quad -\beta \leq y \leq \beta, \quad \lambda(0) = 0.$$

HYPOTHÈSE M. $\gamma(x)$, $\sigma(x)$, $\tau(y)$ et $\lambda(y)$ sont des fonctions de classe C^1 dans les intervalles $[-a, a]$ et $[-\beta, \beta]$ respectivement et telles que $\sigma(0) = \tau(0)$ et que

$$(3) \quad AB < 1, \quad \text{où} \quad A = \max_{-a \leq x \leq a} |\gamma'(x)|, \quad B = \max_{-\beta \leq y \leq \beta} |\lambda'(y)|.$$

THÉORÈME 1. Admettons l'hypothèse M^*) et supposons que la fonction $f(x, y, z, p, q)$, continue dans l'ensemble Π , y satisfait à la condition de Lipschitz

$$(4) \quad |f(x, y, z, p, q) - f(x, y, \tilde{z}, \tilde{p}, \tilde{q})| \leq L\{|z - \tilde{z}| + |p - \tilde{p}| + |q - \tilde{q}|\}.$$

Alors, dans l'hypothèse que la constante k est suffisamment petite, l'équation (1) admet une seule solution $z(x, y)$, continue avec ses dérivées z_x, z_y, z_{xy} dans le rectangle R et satisfaisant aux conditions:

$$(5) \quad z(x, \gamma(x)) = \sigma(x), \quad \text{lorsque} \quad -a \leq x \leq a,$$

$$(6) \quad z(\lambda(y), y) = \tau(y), \quad \text{lorsque} \quad -\beta \leq y \leq \beta.$$

LEMME. Désignons par E_ξ et E_η les espaces vectoriels normés et complets des points ξ et η respectivement. Supposons que $\tilde{\xi} = \Phi(\xi, \eta, \eta) \in E_\xi$ et $\tilde{\eta} = \Psi(\xi, \eta, \xi) \in E_\eta$ et que

$$\|\Phi(\xi, \eta, \eta^*) - \Phi(\xi_1, \eta_1, \eta_1^*)\| \leq c\{\|\xi - \xi_1\| + \|\eta - \eta_1\|\} + A\|\eta^* - \eta_1^*\|,$$

$$\|\Psi(\xi, \eta, \xi^*) - \Psi(\xi_1, \eta_1, \xi_1^*)\| \leq c\{\|\xi - \xi_1\| + \|\eta - \eta_1\|\} + B\|\xi^* - \xi_1^*\|,$$

où

$$(7) \quad AB < 1.$$

*) Un exemple donné dans la Remarque 4 de la note [4] met en évidence l'importance de cette hypothèse.

Alors dans l'hypothèse que la constante c est suffisamment petite, le système

$$(8) \quad \xi = \Phi(\xi, \eta, \eta), \quad \eta = \Psi(\xi, \eta, \xi)$$

n'a qu'une seule solution (ξ, η) , $\xi \in E_\xi$, $\eta \in E_\eta$.

Démonstration. Pour chaque n naturel le système (8) est équivalent au système des deux équations différentielles:

$$\xi = \Phi[\xi, \eta, \Psi(\xi, \eta, \xi)], \quad \eta = \Psi\{\xi, \eta, \Phi[\xi, \eta, \Psi[\xi, \eta, \Phi[\xi, \eta, \Psi[\dots] \dots]] \dots] \}.$$

2n fois

Ce système possède dans l'hypothèse

$$(9) \quad AB + B(AB)^n < 1,$$

en vertu du théorème de Banach *), une seule solution pour chaque c suffisamment petit. L'inégalité (9) résultant de (7) pour n suffisamment grand, notre lemme se trouve ainsi démontré.

Démonstration du théorème 1. On vérifie aisément que pour démontrer le théorème 1, il suffit d'établir l'existence d'une seule solution continue du système

$$(10) \quad v = \Phi[v, w, w], \quad w = \Psi[v, w, v],$$

où les opérateurs $\Phi[v, w, w^*]$ et $\Psi[v, w, v^*]$ sont définis par les formules

$$(11) \quad \begin{aligned} \Phi[v, w, w^*] &= \sigma'(x) - \gamma'(x)w^*(x, \gamma(x)) + \\ &+ \int_{\gamma(x)}^y f[x, t, \tau(t) + \int_{\lambda(t)}^x v(s, t)ds, v(x, t), w(x, t)]dt, \\ \Psi[v, w, v^*] &= \tau'(y) - \lambda'(y)v^*(\lambda(y), y) + \\ &+ \int_{\lambda(y)}^x f[s, y, \tau(y) + \int_{\lambda(y)}^s v(s_1, y)ds_1, v(s, y), w(s, y)]ds. \end{aligned}$$

Si $v(x, y)$, $w(x, y)$ est une solution du système (10) continue dans le rectangle R , alors la fonction

$$z(x, y) = \int_{\lambda(y)}^x v(s, y)ds + \tau(y)$$

est une solution de l'équation (1) de classe C^1 dans le rectangle R qui vérifie les conditions (5) et (6).

Il résulte des relations (2)–(4) que le système (10) satisfait à toutes les hypothèses du lemme précédent dans lequel $c = kL(k+1)$ et E_ξ , E_η désignent les espaces C des fonctions définies dans le rectangle R . Le sys-

*) On applique ce théorème dans l'espace de couples (ξ, η) où $\xi \in E_\xi$, $\eta \in E_\eta$, et la norme est donnée par la formule $\|(\xi, \eta)\| = \|\xi\| + \|\eta\|$.

tème (10) admet donc pour chaque k suffisamment petit une solution unique continue dans le rectangle R . Le théorème 1 se trouve ainsi démontré.

2. THÉORÈME 2. *Admettons l'hypothèse M et supposons que la fonction $f(x, y, z, p, q)$, continue et bornée dans l'ensemble Π , y satisfait à la condition de Lipschitz*

$$|f(x, y, z, p, q) - f(x, y, z, \tilde{p}, \tilde{q})| \leq L \{|p - \tilde{p}| + |q - \tilde{q}|\},$$

et que

$$kL < 1, \quad AB < (1 - kL)^2.$$

Alors l'équation (1) admet une solution $z(x, y)$ continue avec ses dérivées z_x, z_y, z_{xy} dans R et satisfaisant aux conditions (5) et (6).

Démonstration. Désignons par N la borne supérieure (finie par hypothèse) de la fonction $|f(x, y, z, p, q)|$ dans l'ensemble Π et par c et d des nombres finis tels que $|\sigma'(x)| \leq c$ lorsque $-a \leq x \leq a$, $|\tau(y)| \leq d$, $|\tau'(y)| \leq c$ lorsque $-\beta \leq y \leq \beta$. Soient r_1 et r_2 des nombres arbitraires, satisfaisant aux inégalités:

$$(1 - AB)r_1 > (c + Nk)(A + 1), \quad c + Br_1 + Nk \leq r_2 \leq (r_1 - Nk - c)/A.$$

Soit $\delta(\varepsilon)$ le module de continuité ([4], p. 70, renvoi) commun à la fonction $f(x, y, z, p, q)$ dans l'ensemble: $(x, y) \in R$, $|z| \leq d + kr_1$, $|p| \leq r_1$, $|q| \leq r_2$, aux fonctions $\sigma'(x)$, $\gamma'(x)$ dans l'intervalle $[-a, a]$ et aux fonctions $\tau'(y)$, $\lambda'(y)$ dans l'intervalle $[-\beta, \beta]$.

Désignons par $\delta_1(\varepsilon)$ et $\delta_2(\varepsilon)$ des fonctions non-décroissantes, tendant vers zero avec ε :

$$\delta_1(\varepsilon) = (1 + k + r_2)\delta(\varepsilon) + (2A + Lk)N\varepsilon + k\delta(r_1\varepsilon),$$

$$\delta_2(\varepsilon) = (1 + k + r_1)\delta(\varepsilon) + (2B + Lk)N\varepsilon + k\delta(c\varepsilon + Br_1\varepsilon + kN\varepsilon).$$

On vérifie facilement que le système d'équations

$$\delta_1(\varepsilon) + A\mu(A\varepsilon) = (1 - kL)\kappa(\varepsilon), \quad \delta_2(\varepsilon) + B\kappa(B\varepsilon) = (1 - kL)\mu(\varepsilon),$$

possède en vertu de nos hypothèses une seule solution $\kappa(\varepsilon)$, $\mu(\varepsilon)$, définie pour $\varepsilon \geq 0$, non-décroissante et tendant vers zéro avec ε .

Désignons par E l'espace des couples (v, w) , où v et w appartiennent à l'espace C des fonctions définies dans le rectangle R .

Soit $\|(v, w)\| = \|v\| + \|w\|$, $b(v, w) = (bv, bw)$, où $b = \text{const.}$, $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$, où $(v_1, w_1) \in E$, $(v_2, w_2) \in E$.

Pour démontrer le théorème 2 il suffit d'établir l'existence dans l'espace E (linéaire, normé et complet) d'un point fixe de la transformation

$$(12) \quad \tilde{v} = \Phi[v, w, w], \quad \tilde{w} = \Psi[v, w, v],$$

où les opérateurs Φ et Ψ sont donnés par (11). On y parvient en appliquant le théorème de Schauder ([2], Satz 1, p. 173) à la transforma-

tion (12) et à l'ensemble Z (contenu dans l'espace E) des fonctions $v(x, y)$, $w(x, y)$ définies dans le rectangle R et y satisfaisant aux inégalités:

$$\begin{aligned} |v(x, y)| &\leq r_1, & |w(x, y)| &\leq r_2, \\ |v(x, \bar{y}) - v(x, \bar{\bar{y}})| &\leq N|\bar{y} - \bar{\bar{y}}|, & |w(x, \bar{y}) - w(x, \bar{\bar{y}})| &\leq N|\bar{x} - \bar{\bar{x}}|, \\ |v(\bar{x}, y) - v(\bar{\bar{x}}, y)| &\leq \kappa(|\bar{x} - \bar{\bar{x}}|), & |w(x, \bar{y}) - w(x, \bar{\bar{y}})| &\leq \mu(|\bar{y} - \bar{\bar{y}}|). \end{aligned}$$

3. Les théorèmes 1 et 2 restent vrais dans le cas où $\sigma(x)$ et $\tau(y)$ sont des fonctions vectorielles et (1) désigne le système de n équations différentielles

$$z_{xy}^{(i)}(x, y) = f^{(i)}(x, y, z^{(1)}, \dots, z_x^{(n)}, z_y^{(1)}, \dots, z_x^{(n)}, z_y^{(1)}, \dots, z_y^{(n)}) \quad (i=1, \dots, n),$$

écrit sous la forme vectorielle *). Le problème considéré dans ces théorèmes peut être généralisé de la même façon que les problèmes I et II de la note [4] ont été généralisés par les problèmes I* et II* de la note [5]. La méthode appliquée pour démontrer le théorème 2 (théorème 1) permet aussi d'établir l'existence d'une solution (de la solution unique) du problème ainsi généralisé.

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*) La norme d'un vecteur $u = (u_1, \dots, u_n)$ étant donnée par la formule $|u| = \max_{1 \leq i \leq n} |u_i|$.

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Sur un problème concernant un système d'équations différentielles hyperboliques d'ordre arbitraire à deux variables indépendantes

par

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Présenté par T. WAŻEWSKI le 18 Mars, 1957

1. Considérons les courbes continues Γ_i et Λ_j

$$y = \gamma_i(x), \quad \text{lorsque} \quad -a \leq x \leq a \quad (i = 1, \dots, p),$$

$$x = \lambda_j(y), \quad \text{lorsque} \quad -\beta \leq y \leq \beta \quad (j = 1, \dots, m),$$

situées dans le rectangle R défini par les inégalités

$$-a \leq x \leq a \quad \text{et} \quad -\beta \leq y \leq \beta, \quad \text{où} \quad 0 < a < \infty \quad \text{et} \quad 0 < \beta < \infty.$$

Soit

$$(1) \quad k = 2 \max(a, \beta).$$

Supposant que $u(x, y) = (u^{(1)}(x, y), \dots, u^{(n)}(x, y))$ est une fonction définie dans le rectangle R et y est différentiable m fois par rapport à x ($m \geq 1$) et p fois par rapport à y ($p \geq 1$), désignons par V , W et Z les fonctions suivantes:

$$V(x, y) = (v_1(x, y), \dots, v_p(x, y)), \quad \text{où} \quad v_i(x, y) = \frac{\partial^{m+i-1} u(x, y)}{\partial x^m \partial y^{i-1}} \quad (i = 1, \dots, p),$$

$$W(x, y) = (w_1(x, y), \dots, w_m(x, y)), \quad \text{où} \quad w_j(x, y) = \frac{\partial^{p+j-1} u(x, y)}{\partial y^p \partial x^{j-1}} \quad (j = 1, \dots, m),$$

$$(3) \quad Z(x, y) = (\dots, z_{ij}(x, y), \dots), \quad \text{où} \quad z_{ij}(x, y) = \frac{\partial^{i+j-2} u(x, y)}{\partial x^{j-1} \partial y^{i-1}} \quad (i = 1, \dots, p; j = 1, \dots, m).$$

PROBLÈME Q. Existe-t-il une solution $u(x, y)$ du système

$$(4) \quad \frac{\partial^{p+m} u(x, y)}{\partial x^m \partial y^p} = f(x, y, Z(x, y), V(x, y), W(x, y))$$

(au second membre continu) remplissant les conditions suivantes:

C₁) La fonction $u(x, y)$ ainsi que toutes ses dérivées qui interviennent dans l'équation (4) sont continues dans le rectangle R .

C₂) $v_i(x, y) = g_i(x, Z(x, y), W(x, y))$, lorsque $y = \gamma_i(x)$, $-\alpha \leq x \leq \alpha$,
 $w_j(x, y) = h_j(y, Z(x, y), V(x, y))$, lorsque $x = \lambda_j(y)$, $-\beta \leq y \leq \beta$,

où g_i ($i=1, \dots, p$) et h_j ($j=1, \dots, m$) sont des fonctions continues données d'avance.

C₃) $z_{ij}(x, y) = \hat{u}_{ij}$, lorsque $(x, y) = (\hat{x}_{ij}, \hat{y}_{ij})$,

où $(\hat{x}_{ij}, \hat{y}_{ij})$ sont des points arbitrairement choisis dans le rectangle R , et \hat{u}_{ij} sont des vecteurs donnés ($i=1, \dots, p$; $j=1, \dots, m$).

Remarque 1. Le problème Q généralise le problème I de la note [2] de la même façon que le problème de Nicoletti généralise celui de Darboux. D'une manière analogue on peut généraliser le problème I* de la note [3].

2. LEMME. Il existe une correspondance biunivoque entre les solutions du problème Q et les solutions $(V(x, y), W(x, y))$ continues dans le rectangle R du système S,

$$\begin{aligned}
 & \left. \begin{aligned}
 v_i(x, y) &= g_i\{x, L[V(\cdot), W(\cdot), x, \gamma_i(x)], W(x, \gamma_i(x))\} + \int_{\gamma_i(x)}^y v_{i+1}(x, t) dt \\
 & \qquad \qquad \qquad (i=1, \dots, p-1), \\
 v_p(x, y) &= g_p\{x, L[V(\cdot), W(\cdot), x, \gamma_p(x)], W(x, \gamma_p(x))\} + \\
 & \qquad \qquad \qquad + \int_{\gamma_p(x)}^y f\{x, t, L[V(\cdot), W(\cdot), x, t], V(x, t), W(x, t)\} dt, \\
 w_j(x, y) &= h_j\{y, L[V(\cdot), W(\cdot), \lambda_j(y), y], V(\lambda_j(y), y)\} + \int_{\lambda_j(y)}^x w_{j+1}(s, y) ds \\
 & \qquad \qquad \qquad (j=1, \dots, m-1), \\
 w_m(x, y) &= h_m\{y, L[V(\cdot), W(\cdot), \lambda_p(y), y], V(\lambda_p(y), y)\} + \\
 & \qquad \qquad \qquad + \int_{\lambda_m(y)}^x f\{s, y, L[V(\cdot), W(\cdot), s, y], V(s, y), W(s, y)\} ds,
 \end{aligned} \right\} S:
 \end{aligned}$$

dans lequel $L[V(\cdot), W(\cdot), x, y]$ est un opérateur convenablement choisi, défini pour $(x, y) \in R$ et toutes les fonctions $V(x, y)$, $W(x, y)$ continues dans le rectangle R et qui possède les propriétés suivantes *):

P₁) $|L[V(\cdot), W(\cdot), x, y] - L[\tilde{V}(\cdot), \tilde{W}(\cdot), x, y]| \leq \chi(k) \{\|V - \tilde{V}\| + \|W - \tilde{W}\|\}$,
 pour tous $(x, y) \in R$, où $\chi(k) \rightarrow 0$, lorsque $k \rightarrow 0$ (cf. (1)).

*) Les valeurs de l'opérateur L sont des vecteurs. Si $\psi(x, y) = (\psi^{(1)}(x, y), \dots, \psi^{(s)}(x, y))$ est une fonction vectorielle continue dans R , alors

$$|\psi(x, y)| = \max_{1 \leq i \leq s} |\psi^{(i)}(x, y)|, \quad \|\psi(x, y)\| = \max_{(x, y) \in R} |\psi(x, y)|.$$

P_2) Dans l'hypothèse $K: \|V(x, y)\| \leq r, \|W(x, y)\| \leq r, 0 < r < \infty, \|V(x, \tilde{y}) - V(x, y)\| \leq \mu(|\tilde{y} - y|), \|W(\tilde{x}, y) - W(x, y)\| \leq \mu(|\tilde{x} - x|)$, où $\mu(\varepsilon)$ est une fonction non-décroissante qui tend vers zéro avec ε , on a $|L| \leq \max_{1 \leq i \leq p} |\dot{u}_{ij}| + \chi_1(k)r$, où $\chi_1(k) \rightarrow 0$ lorsque $k \rightarrow 0$, $|L[V(\cdot), W(\cdot), \tilde{x}, \tilde{y}] - L[V(\cdot), W(\cdot), x, y]| \leq \varrho(|\tilde{x} - x|) + \varrho(|\tilde{y} - y|)$ pour tous $(x, y) \in R, (\tilde{x}, \tilde{y}) \in R$, où $\varrho(\varepsilon)$ est une fonction non-décroissante, ne dépendant pas du choix particulier des fonctions V et W satisfaisant à l'hypothèse K , et telle que $\varrho(\varepsilon) \rightarrow 0$ lorsque $\varepsilon \rightarrow 0$ ($\varrho(\varepsilon)$ dépend de $\mu(\varepsilon)$).

Démonstration. Soit $\tilde{u}(x, y)$ une solution du problème Q . Désignons par \tilde{V}, \tilde{W} et \tilde{Z} les dérivées de la fonction \tilde{u} , définies par (2) et (3). Il existe alors l'opérateur $L[V(\cdot), W(\cdot), x, y]$ ayant les propriétés P_1, P_2 et tel que *) $\tilde{Z}(x, y) = L[\tilde{V}(\cdot), \tilde{W}(\cdot), x, y]$. Il en résulte que la fonction $(\tilde{V}(x, y), \tilde{W}(x, y))$ est une solution continue du système S . Elle correspond à $\tilde{u}(x, y)$, ce que nous notons de manière suivante: $(\tilde{V}, \tilde{W}) = \Delta(\tilde{u})$. Supposons maintenant que $(V^*(x, y), W^*(x, y))$ soit une solution continue du système S dans lequel L désigne l'opérateur défini ci-dessus. Soit $Z^*(x, y) = (\dots, z_{ij}^*(x, y), \dots) = L[V^*(\cdot), W^*(\cdot), x, y]$. On démontre facilement que $z_{11}^*(x, y)$ est une solution du problème Q et que $(V^*, W^*) = \Delta(z_{11}^*)$.

3. Existence locale de la solution unique du problème Q .

HYPOTHÈSE H . 1° La fonction $f(x, y, Z, V, W)$ continue dans l'ensemble $\Pi: -a \leq x \leq a$ et $-\beta \leq y \leq \beta$ (avec Z, V, W arbitraires) y satisfait à la condition de Lipschitz par rapport aux variables Z, V, W .

2° Les fonctions $g_i(x, Z, W)$ ($i = 1, \dots, p$) et $h_j(y, Z, V)$ ($j = 1, \dots, m$) continues dans les ensembles $\Sigma_1: -a \leq x \leq a$ (avec Z, W arbitraires) et $\Sigma_2: -\beta \leq y \leq \beta$ (avec Z, V arbitraires) respectivement y satisfont à la condition de Lipschitz, à savoir:

$$|g_i(x, Z, W) - g_i(x, \tilde{Z}, \tilde{W})| \leq c_1 |Z - \tilde{Z}| + A |W - \tilde{W}|,$$

$$|h_j(y, Z, V) - h_j(y, \tilde{Z}, \tilde{V})| \leq c_2 |Z - \tilde{Z}| + B |V - \tilde{V}|,$$

où $AB < 1$.

THÉORÈME 1. Dans l'hypothèse H et celle que k (cf. (1)) est suffisamment petit, le problème Q a une seule solution.

La démonstration du théorème 1, étant analogue à celle du théorème 1 de la note [4], nous nous bornerons à l'esquisser. Il est évident qu'en vertu de notre lemme il suffit d'établir l'existence et l'unicité d'une

*) L'opérateur L est défini pour $(x, y) \in R$ et toutes les fonctions $V(x, y), W(x, y)$ continues dans R . Dans le cas $p = m = 1$ on a $V = v_1, W = w_1$ et $L[V(\cdot), W(\cdot), x, y] = \dot{u}_{11} + \int_{z_{11}}^x v_1(s, \dot{y}_{11}) ds + \int_{y_{11}}^y w_1(x, t) dt$.

solution continue $(V(x, y), W(x, y))$ du système S dans lequel l'opérateur L jouit des propriétés P_1 et P_2 . On le vérifie aisément en tenant compte de l'hypothèse H et en utilisant le lemme de la note [4] dans lequel $\xi = V(x, y)$, $\eta = W(x, y)$ et $c = \chi^*(k)$, où $\chi^*(k)$ est une fonction convenablement choisie tendant vers zéro avec k (cf. (1)).

Remarque 2. On peut remplacer dans le théorème 1 les ensembles Π , Σ_1 et Σ_2 par les ensembles:

$$\Pi^*: -a \leq x \leq a, \quad -\beta \leq y \leq \beta, \quad |Z| \leq r, \quad |V| \leq r, \quad |W| \leq r, \quad (0 < r < \infty),$$

$$\Sigma_1^*: -a \leq x \leq a, \quad |Z| \leq r, \quad |W| \leq r,$$

$$\Sigma_2^*: -\beta \leq y \leq \beta, \quad |Z| \leq r, \quad |V| \leq r,$$

respectivement, admettant supplémentairement que $|g_i(x, Z, W)| < r$ dans l'ensemble Σ_1^* , $|h_j(y, Z, V)| < r$ dans l'ensemble Σ_2^* et que $|\hat{u}_{ij}| < r$ ($i = 1, \dots, p$; $j = 1, \dots, m$).

4. Existence d'une solution (pas forcément unique) du problème Q .

HYPOTHÈSE M. Les fonctions $g_i(x, Z, W)$ ($i = 1, \dots, p$) et $h_j(y, Z, V)$ ($j = 1, \dots, m$) continues dans les ensembles Σ_1^* et Σ_2^* respectivement y satisfont à des conditions:

$$|g_i(x, Z, W) - g_i(x, Z, \tilde{W})| \leq A|W - \tilde{W}|, \quad |g_i(x, Z, W)| < r,$$

$$|h_j(y, Z, V) - h_j(y, Z, \tilde{V})| \leq B|V - \tilde{V}|, \quad |h_j(y, Z, V)| < r,$$

où $AB < 1$.

THÉORÈME 2. Admettons l'hypothèse M et supposons que la fonction $f(x, y, Z, V, W)$ continue dans l'ensemble Π^* y satisfait à la condition de Lipschitz par rapport aux variables V et W et que les constantes k (cf. (1)) et $|\hat{u}_{ij}|$ ($i = 1, \dots, p$; $j = 1, \dots, m$) sont suffisamment petites.

Alors le problème Q admet une solution.

Le théorème 2 résulte aisément de notre lemme et du théorème 3, énoncé ci-dessous *), qui concerne le système d'équations:

$$(5) \quad \left\{ \begin{array}{l} v_i(x, y) = \int_{\gamma_i(x)}^y \tau_i(x, t, y, L_i[V(\cdot), W(\cdot), x, t], V(x, t), W(x, t)) dt + \\ \quad + g_i(x, \mathcal{L}_i[V(\cdot), W(\cdot), x], W(x, \gamma_i(x))) \quad (i = 1, \dots, p), \\ w_j(x, y) = \int_{\lambda_j(y)}^x \xi_j(s, x, y, \hat{L}_j[V(\cdot), W(\cdot), s, y], V(s, y), W(s, y)) ds + \\ \quad + h_j(y, \hat{\mathcal{L}}_j[V(\cdot), W(\cdot), y], V(\lambda_j(y), y)) \quad (j = 1, \dots, m), \end{array} \right.$$

*) Le théorème 3 est énoncé dans une forme plus générale que celle dont on aurait besoin pour en déduire le théorème 2.

dans lequel L_i, \mathcal{L}_i ($i=1, \dots, p$), $\hat{L}_j, \hat{\mathcal{L}}_j$ ($j=1, \dots, m$) désignent des opérateurs qui sont continus par rapport à V et W ($x, s \in [-a, a]$, $y, t \in [-\beta, \beta]$ et $V, W \in$ l'espace C des fonctions définies dans R) et jouissent de la propriété suivante: Dans l'hypothèse K (cf. lemme, propriété P_2), les valeurs des opérateurs $L_i, \mathcal{L}_i, \hat{L}_j, \hat{\mathcal{L}}_j$ sont bornées par r et il existe une fonction non-décroissante $\varrho(\varepsilon)$, $\varrho(\varepsilon) \rightarrow 0$ lorsque $\varepsilon \rightarrow 0$, telle que

$$\begin{aligned}
 |L_i[V(\cdot), W(\cdot), \tilde{x}, \tilde{y}] - L_i[V(\cdot), W(\cdot), x, y]| &\leq \varrho(|\tilde{x} - x|) + \varrho(|\tilde{y} - y|), & (i=1, \dots, p), \\
 |\mathcal{L}_i[V(\cdot), W(\cdot), \tilde{x}] - \mathcal{L}_i[V(\cdot), W(\cdot), x]| &\leq \varrho(|\tilde{x} - x|), \\
 |\hat{L}_j[V(\cdot), W(\cdot), \tilde{x}, \tilde{y}] - \hat{L}_j[V(\cdot), W(\cdot), x, y]| &\leq \varrho(|\tilde{x} - x|) + \varrho(|\tilde{y} - y|), & (j=1, \dots, m), \\
 |\hat{\mathcal{L}}_j[V(\cdot), W(\cdot), \tilde{y}] - \hat{\mathcal{L}}_j[V(\cdot), W(\cdot), y]| &\leq \varrho(|\tilde{y} - y|),
 \end{aligned}$$

pour tous $(x, y) \in R$, $(\tilde{x}, \tilde{y}) \in R$.

THÉORÈME 3. *Admettons l'hypothèse M et supposons que les fonctions $\tau_i(x, t, y, Z_i, V, W)$ ($i=1, \dots, p$) et $\zeta_j(s, x, y, \hat{Z}_j, V, W)$ ($j=1, \dots, m$) sont continues dans les ensembles A_i : $-a \leq x \leq a$, $-\beta \leq t \leq \beta$, $-\beta \leq y \leq \beta$, $|Z_i| \leq r$, $|V| \leq r$, $|W| \leq r$ et B_j : $-a \leq s \leq a$, $-a \leq x \leq a$, $-\beta \leq y \leq \beta$, $|\hat{Z}_j| \leq r$, $|V| \leq r$, $|W| \leq r$ respectivement, et y satisfont à des conditions de Lipschitz par rapport aux variables V et W .*

Alors dans l'hypothèse que la constante k (cf. (1)) est suffisamment petite, le système (5) admet une solution continue.

La démonstration du théorème 3, au fond analogue à celle du théorème 2 de la note [3], sera omise ici.

Remarque 3. Il est évident que le théorème 3 permet d'établir l'existence d'une solution du système

$$\begin{cases} \frac{\partial v_i(x, y)}{\partial y} = \tau_i^* \{x, y, L_i[V(\cdot), W(\cdot), x, y], V(x, y), W(x, y)\} & (i=1, \dots, p), \\ \frac{\partial w_j(x, y)}{\partial x} = \zeta_j^* \{x, y, \hat{L}_j[V(\cdot), W(\cdot), x, y], V(x, y), W(x, y)\} & (j=1, \dots, m), \end{cases}$$

remplissant les conditions

$$\begin{aligned}
 v_i(x, \gamma_i(x)) &= g_i \{x, \mathcal{L}_i[V(\cdot), W(\cdot), x], W(x, \gamma_i(x))\} & (i=1, \dots, p), \\
 w_j(\lambda_j(y), y) &= h_j \{y, \hat{\mathcal{L}}_j[V(\cdot), W(\cdot), y], V(\lambda_j(y), y)\} & (j=1, \dots, m).
 \end{aligned}$$

5. Existence intégrale de la solution du problème \bar{Q} .

Remarque 4. Le théorème 1 (ayant le caractère local) peut quelquefois servir à démontrer les théorèmes dans lesquels la constante k n'est soumise à aucune restriction. En effet, dans certaines hypothèses supplémentaires sur les courbes Γ_i ($i=1, \dots, p$) et A_j ($j=1, \dots, m$), le rectangle R se laisse diviser par des parallèles aux axes des coordonnées en rectangles R_i ($i=1, \dots, s$), ayant cette propriété que la solution du

problème Q dans le rectangle R tout entier peut être obtenue par le prolongement des solutions des problèmes respectifs du rectangle R_i à R_{i+1} ($i=1, \dots, s-1$) *).

Remarque 5. L'hypothèse que la constante k est suffisamment petite peut être remplacée dans le théorème 1 par celle que les constantes de Lipschitz sont suffisamment petites (cf. hypothèse H) et dans le théorème 2 par l'hypothèse que la constante de Lipschitz de la fonction f par rapport aux variables V et W ainsi que le produit des constantes $A \cdot B$ (cf. hypothèse M) et les bornes supérieures des fonctions f, g, i ($i=1, \dots, p$), h_j ($j=1, \dots, m$) sont suffisamment petites.

On établit ce résultat en modifiant les démonstrations des théorèmes 1 et 2, à savoir on remplace le système S par un système \tilde{S} équivalent convenablement choisi **).

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OUVRAGES CITÉS

[1] G. Stampacchia, *Il problema di Goursat per un'equazione alle derivate parziali del secondo ordine di tipo iperbolico*, Giornale di Matematiche di Battaglini, s. IV, **79** (1949/50), 66.

[2] Z. Szmydt, *Sur un nouveau type de problèmes pour un système d'équations différentielles hyperboliques du second ordre à deux variables indépendantes*, Bull. Acad. Polon. Sci., Cl. III, **4** (1956), 67.

[3] — *Sur une généralisation des problèmes classiques concernant un système d'équations différentielles hyperboliques du second ordre à deux variables indépendantes*, Bull. Acad. Polon. Sci., Cl. III, **4** (1956), 579.

[4] — *Sur le problème de Goursat concernant les équations différentielles hyperboliques du second ordre*, Bull. Acad. Polon. Sci., Cl. III, **5**, (1957), 571.

*) Cette méthode a été appliquée par G. Stampacchia ([1] p. 81-85) dans la démonstration de l'existence globale de la solution unique du problème de Goursat.

**) Soit S_1 le système englobant les p premières équations du système S et S_2 celui qui englobe les m dernières équations de S . Pour obtenir le système \tilde{S} on transforme les systèmes S_1 et S_2 en systèmes équivalents \tilde{S}_1 et \tilde{S}_2 en utilisant un procédé que nous expliquerons dans le cas du système S_1 : l'équation p -ième reste invariante; on remplace v_p intervenant sous le signe de l'intégrale dans l'équation $p-1$ -ième par le second membre de l'équation p -ième; dans le système d'équations ainsi obtenu on substitue le second membre de l'équation $p-1$ -ième au v_{p-1} intervenant sous le signe de l'intégrale dans l'équation $p-2$ -ième, etc. Au bout de $p-1$ pas on arrive au système \tilde{S}_1 .

One-Sided Non-Uniqueness in Ordinary Differential Equations

by

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Presented by T. WAŻEWSKI on March 18, 1957

In this note we shall give an example of a system of two differential equations of the form $x'_1 = f_1(t, x_1, x_2)$, $x'_2 = f_2(t, x_1, x_2)$, i. e., (in the vector notation) of the form

$$(1) \quad X' = F(t, X),$$

where the vector function $F(t, X)$, $F = (f_1, f_2)$, $X = (x_1, x_2)$ is continuous and bounded in the space R^3 ; for this system each point of the space R^3 is a point of left-sided uniqueness (see the definition below) and also a point of right-sided non-uniqueness *).

The problem of the existence of an example of this kind has been posed by C. Ryll-Nardzewski. Such an example cannot be constructed in the case of one equation ([1], Théorème VII). Now, M. Lavrentieff [2] has given the classical example of one equation, $x' = f(t, x)$, with the right side continuous in an open square, each point of which is a point of (left- or right-sided) non-uniqueness.

Notations and definitions. Capital Roman letters, with the exception of P , will denote points of the space R^2 , and the letter P will denote the point (t, X) of the space R^3 . Moreover, we shall assume that:

$$|X| = \sqrt{x_1^2 + x_2^2}, \quad |P| = \max(|t|, |X|)$$

$c(s; P; F(-)) = \max_{Y(-) \in A, Z(-) \in A} |Y(s) - Z(s)|$, where A denotes the family of all the integrals of Eq. (1) passing through the point P .

$$d(t, X; F(-)) = \max_{t-1 \leq s \leq t} c(s; t, X; F(-)).$$

*) This result was communicated at the meeting of the Section of Differential Equations, Mathematical Institute, Polish Academy of Sciences, in November 1952.

The point (t, X) is called a point of right-sided non-uniqueness for Eq. (1) if for any positive number r there exists a number s , $t < s < t + r$, such that $c(s; t, X; F(-)) > 0$. Otherwise, the point (t, X) is called a point of right-sided uniqueness. Points of left-sided uniqueness and non-uniqueness are defined in an analogical way.

$J(s; P; F(-))$ will denote an integral of Eq. (1) passing through the point P .

LEMMA 1. For numbers $\alpha, \beta, \alpha < \beta$ and a system of such points $A^1, \dots, A^q, B^1, \dots, B^q$ of the circle $\Phi: |X| < 1$ that $A^i \neq A^j, B^i \neq B^j$ ($i, j = 1, \dots, q, i \neq j$) $|A^i - B^i| < \gamma$ ($i = 1, \dots, q$) there exists such a function $G(t, X)$ of class C^∞ in the cylinder $\Lambda_{(\alpha, \beta)}: \alpha \leq t \leq \beta, |X| \leq 1$ that $|G(t, X)| = 0$ in a certain neighbourhood of the boundary $\Lambda_{(\alpha, \beta)}, |G(t, X)| < 2\gamma/(\beta - \alpha)$ in $\Lambda_{(\alpha, \beta)}$,

$$(2) \quad J(\beta; \alpha, A^i; G(-)) = B^i \quad (i = 1, \dots, q):$$

Proof. Let $\varphi(s), \psi(s)$ be such functions of class C^∞ on a straight line that $\varphi(s) = 0$ for $s \leq \frac{1}{5}$ and for $s \geq \frac{4}{5}$, $0 < \varphi(s) \leq 1$ for $\frac{1}{5} < s < \frac{4}{5}$, $\varphi(\frac{1}{2}) = 1$, $\psi(s) = 0$ for $s \leq \frac{1}{5}$, $\psi(s) = 1$ for $s \geq \frac{4}{5}$, $0 \leq \psi(s) \leq 1$, $0 \leq \psi'(s) < 2$ for any s .

The vector functions $U^i(t) = A^i + \psi(t - \alpha/(\beta - \alpha))(B^i - A^i) + C^i \varphi(t - \alpha/(\beta - \alpha))$ (where C^i are given points) ($i = 1, \dots, q$) are of class C^∞ and satisfy the relations $U^i(\alpha) = A^i, U^i(\beta) = B^i$. For $|C^i| \leq \varepsilon$ ($i = 1, \dots, q$), where ε is a sufficiently small positive number, we have

$$\left| \frac{dU^i(t)}{dt} \right| < \frac{2\gamma}{\beta - \alpha}, \quad |U^i(t)| < 1 \quad (i = 1, \dots, q).$$

Let us fix the system of points C^1, \dots, C^q so that $|C^i| \leq \varepsilon$ and that the curves $X = U^i(t)$ be disjoint. Let us fix a positive number r so that the sets $\Pi^i: |X - U^i(t)|^2 < r$ be also disjoint and contained in the set $|X| < 1$.

Assume

$$G(t, X) = \sum_{i=1}^q \frac{dU^i(t)}{dt} \varphi\left(\frac{|X - U^i(t)|^2}{r} + \frac{1}{2}\right).$$

It can easily be seen that the curves $X = U^i(t)$ ($i = 1, \dots, q$) are integrals of the system $X' = G(t, X)$ and the function $G(t, X)$ satisfies Lemma 1.

LEMMA 2. For any positive numbers ε, M and any numbers $\tau, \sigma, \tau < \sigma$ there exist a positive number δ and a vector function $H(t, X) = (h_1(t, X), h_2(t, X))$ continuous in R^3 , of class C^∞ for $t \neq \tau, |H(t, X)| = 0$ for $t \leq \tau$, for $t \geq \sigma$ and for $|X| \geq M, |H(t, X)| \leq \varepsilon$ in R^3 and such that $c(\sigma; \tau, X; H(-)) \geq \delta$ for $|X| \leq M$.

Proof. We shall prove the lemma for $\tau = 0, \sigma = 1, M = 1$. The general case is reducible to the above case by the transformation $\tilde{t} =$

$$= \frac{t - \tau}{\sigma - \tau}, \quad \tilde{X} = \frac{X}{M}.$$

Denote by $S_1^i, \dots, S_{m_i}^i, V_1^i, \dots, V_{m_i}^i$ ($0 \leq i$) the system of distinct points of the circle Φ such that for every point X of the circle Φ there exist points S_j^i and V_k^i such that

$$(3) \quad |X - S_j^i| < 4^{-i-1}\varepsilon, \quad |X - V_k^i| < 4^{-i-1}\varepsilon.$$

Let Γ be an open subset of the circle Φ containing the points $S_1^0, \dots, S_{m_0}^0$ and let Δ be an open subset of the circle Φ containing the points $V_1^0, \dots, V_{m_0}^0$, the distance of these sets being positive:

$$(4) \quad \varrho(\Gamma, \Delta) = \delta > 0.$$

We shall construct the function $H(t, X)$ in such a way that

$$(5) \quad J(1; 2^{-i}, S_j^i; H(-)) \in \Gamma, \quad J(1; 2^{-i}, V_j^i; H(-)) \in \Delta \\ (i = 0, 1, \dots; j = 1, \dots, m_i).$$

Assume $H(t, X) = 0$ for $t \geq 1$ and for $|X| > 1$. By the definition of Γ, Δ relations (5) are satisfied for $i = 0$. Now assume that the function $H(t, X)$ has been defined for $t \geq 2^{-(n-1)}$ ($n \geq 1$) and that relations (5) are satisfied for $i = n - 1$. By the first inequality of (3) ($X = S_p^n, i = n - 1$), there exist indices $j(1), \dots, j(m_n)$ such that $|S_p^n - S_{j(p)}^{n-1}| < 4^{-n}\varepsilon$ for $1 \leq p \leq m_n$. In view of the assumption that relations (5) are satisfied for $i = n - 1$, of the continuity of the function $J(1, 2^{-(n-1)}, Z; H(-))$ with respect to Z and of the fact that the set Γ is open, we can assign distinct points Z_p^n to the points S_p^n ($p = 1, \dots, m_n$) in such a way that $|S_p^n - Z_p^n| < 4^{-n}\varepsilon$ and $J(1; 2^{-(n-1)}, Z_p^n; H(-)) \in \Gamma$ ($p = 1, \dots, m_n$).

Analogically, we assign distinct points W_p^n (and different from the points Z_p^n) to the points V_p^n in such a way that $|V_p^n - W_p^n| < 4^{-n}\varepsilon$ and $J(1; 2^{-(n-1)}, W_p^n; H(-)) \in \Delta$ ($p = 1, \dots, m_n$). We apply lemma 1 ($q = 2m_n$) to the numbers $\alpha = 2^{-n}$, $\beta = 2^{-(n-1)}$, to the points $A^i: A^i = S_i^n, A^{i+m_n} = V_i^n$ and to the points $B^i: B^i = Z_i^n, B^{i+m_n} = W_i^n$ ($i = 1, \dots, m_n$). We set $H(t, X) = G(t, X)$ for $2^{-n} \leq t < 2^{-(n-1)}, |X| \leq 1$. By (2) we have $J(1; 2^{-n}, S_j^n; H(-)) = J(1; 2^{-(n-1)}, Z_j^n; H(-))$ and $J(1, 2^{-n}, V_j^n; H(-)) = J(1; 2^{-(n-1)}, W_j^n; H(-))$. Relations (5) are thus satisfied for $i = n$. The function $H(t, X)$ is of class C^∞ for $t \geq 2^{-n}$. Moreover,

$$(6) \quad |H(t, X)| < \frac{2 \cdot 4^{-n} \cdot \varepsilon}{2^{-n}} = 2^{1-n}\varepsilon \quad \text{for} \quad 2^{-n} \leq t < 2^{-(n-1)} (n \geq 1).$$

In this way we define inductively the function $H(t, X)$ for $t > 0$.

Assume $H(t, X) = 0$ for $t \leq 0$. The function $H(t, X)$ is of class C^∞ for $t \neq 0$ and by (6) it is continuous and satisfies the inequality $|H(t, X)| \leq \varepsilon$ in R^3 . Let X be an arbitrary point of the circle $|X| \leq 1$. By (3) there exist a sequence $S_{j_n}^n \rightarrow X$ for $n \rightarrow \infty$ and a sequence $V_{k_n}^n \rightarrow X$ for $n \rightarrow \infty$. From

the sequence of integrals $J(t; 2^{-n}, S_{j_n}^n; H(-))$ we can select a uniformly convergent sequence. The limit of this sequence, $S(t)$, is an integral of the equation $X' = H(t, X)$ and satisfies the condition $S(0) = X$, and, by (5), $S(1) \in \bar{\Gamma}$. Analogically we define the integral $V(t)$ satisfying the conditions $V(0) = X, V(1) \in \bar{\Delta}$. Thus by (4) we have $c(1; 0, X; H(-)) \geq \delta$ for $|X| \leq 1$.

LEMMA 3. If the function $K(t, X)$ is continuous in R^3 ,

$$(7) \quad |K(t, X)| = 0 \quad \text{for} \quad |X| \geq M, \quad c(s; t, X; K(-)) = 0$$

$$\text{for} \quad \tau < t \leq \sigma, \quad t \leq s \leq \sigma, \quad c(\sigma; \tau, X, K(-)) \geq \delta > 0 \quad \text{for} \quad |X| \leq M,$$

then there exists a function $\eta(t)$, continuous for any t , positive for $t \neq \tau$ such that for any function $E(t, X)$ continuous in R^3 and such that

$$|E(t, X) - K(t, X)| \leq \eta(t) \text{ in } R^3$$

we have $c(\sigma; \tau, X; E(-)) > 0$ for $|X| \leq M$.

Proof. We shall assume in the proof that $\tau = 0, \sigma = 1$. We shall prove by induction that there exist positive constants η_0, η_1, \dots and $\varepsilon_0, \varepsilon_1, \dots$ for which the following property Ω holds:

If $W(t)$ satisfies the equation

$$(8) \quad X' = E(t, X),$$

$$(9) \quad |E(t, X) - K(t, X)| < \eta_j \quad \text{for} \quad 2^{-j} \leq t \leq 2^{1-j} \quad \text{and for any } X \\ (j = 1, 2, \dots), \quad \text{and} \quad |U - W(2^{-i})| \leq \varepsilon_i$$

then $|W(1) - J(1; 2^{-i}, U; K(-))| \leq \delta/3$ ($i = 0, 1, \dots$). For $i = 0$ it is sufficient to assume $\varepsilon_0 = \delta/3, \eta_0 = 1$. If the constants $\varepsilon_{k-1}, \eta_{k-1}$ ($k \geq 1$) have already been defined, it is sufficient to define ε_k, η_k in such a way that for an arbitrary integral $V(t)$ of Eq. (8) satisfying (9) for $j = k$ we have the inequality $|V(2^{1-k}) - J(2^{1-k}; 2^{-k}, U; K(-))| \leq \varepsilon_{k-1}$ for $|U - V(2^{-k})| \leq \varepsilon_k$. By (7) the existence of such constants ε_k, η_k can easily be proved indirectly.

Let X be any point of the circle $|X| \leq M$ and let $Y(t), Z(t)$ be such integrals of the equation $X' = K(t, X)$ that $Y(0) = Z(0) = X, |Y(1) - Z(1)| \geq \delta$. For Eq. (8), where $E(t, X)$ satisfies (9), we shall consider the sequence of integrals $A^n(t)$ passing through the points $(2^{-n}, Y(2^{-n}))$ and the sequence of integrals $B^n(t)$ passing through the points $(2^{-n}, Z(2^{-n}))$. The limits, $A(t)$ and $B(t)$, of the sequences suitably picked from the sequences $A^n(t), B^n(t)$ satisfy Eq. (8) and conditions $A(0) = B(0) = X$. Property Ω implies the inequalities $|A(1) - Y(1)| \leq \delta/3, |B(1) - Z(1)| \leq \delta/3$, whence $|A(1) - B(1)| \geq \delta/3$. It suffices to take as the function $\eta(t)$ a continuous function positive for $t \neq 0$ and satisfying the conditions $|\eta(t)| \leq \eta_i$ for $2^{-i} \leq t \leq 2^{1-i}$ ($i = 1, 2, \dots$).

LEMMA 4. For any positive number ξ and a function $G(t, X)$ continuous in R^3 and of class C^∞ in the neighbourhood of the layer $\tau \leq t \leq \sigma$ and such that $|G(t, X)| = 0$ for $|X| \geq M$, $d(P; G(-)) = 0$ in R^3 there exist a function $\gamma(t)$ continuous for any t and positive for $t \neq \tau$ and a function $K(t, X)$ continuous in R^3 , of class C^∞ in the neighbourhood of the layer $\tau < t \leq \sigma$ and such that $d(P; K(-)) = 0$ in R^3 ,

$$(10) \quad |K(t, X)| = 0 \quad \text{for} \quad |X| \geq M,$$

$$(11) \quad K(P) = G(P) \quad \text{for} \quad t \leq \tau \quad \text{and for} \quad t \geq \sigma,$$

$|K(P) - G(P)| \leq \xi$ in R^3 and, moreover, such that for any continuous functions $E(P)$ for which $|E(t, X) - K(t, X)| \leq \gamma(t)$ in R^3 we have the relations $d(P, E(-)) \leq \xi$ for $|P| \leq M$ and $c(\sigma; \tau, X; E(-)) > 0$ for $|X| \leq M$.

Proof. Write $L(t, Y) = J(t; \tau, Y; G(-))$. Let $|\partial L(t, Y)/\partial y_j| < \lambda$ for $j = 1, 2$, $|Y| \leq M$. Apply lemma 2 to $\varepsilon = \xi/2\lambda$ and define the function $K(t, X)$ by the formula

$$K(t, X) = G(t, X) + \left(\sum_{j=1}^2 \frac{\partial L(t, Y)}{\partial y_j} h_j(t, Y) \right) \Big|_{Y=L^{-1}(t, X)},$$

where $L^{-1}(t, X) = J(\tau; t, X; G(-))$.

We have $|K(t, X) - G(t, X)| \leq 2\lambda\varepsilon = \xi$ in R^3 . It is also easy to ascertain (10), (11). If $Y = Y(t)$ satisfies the equation $Y' = H(t, Y)$, then $X = L(t, Y(t))$ satisfies the equation $X' = K(t, X)$. It is thus easy to observe that for $|Y| \leq M$ we have $c(\sigma; \tau, Y; K(-)) \geq \delta/2\gamma > 0$, where $\gamma > |\partial L^{-1}(t, X)/\partial x_j|$ for $j = 1, 2$, $\tau \leq t \leq \sigma$, $|X| \leq M$. It is also easy to prove that the function $K(t, X)$ is continuous in R^3 and of class C^∞ in the neighbourhood of the layer $\tau < t \leq \sigma$, i. e., by (11) and the relation $d(P; G(-)) = 0$, $d(P; K(-)) = 0$ in R^3 . In view of the fact that $d(P; E(-))$ is half-continuous from above with respect to $E(-)$ there exists a positive number ζ such that from $|E(P) - K(P)| \leq \zeta$ results $d(P; E(-)) \leq \xi$ for $|P| \leq M$. Applying lemma 3 and setting $\gamma(t) = \min(\xi, \eta(t))$ we complete the proof of lemma 4.

We now proceed to the construction of the right side of Eq. (1). Let t_i be a sequence of distinct numbers everywhere dense on a straight line. Let us choose the sequence s_i in such a way that $s_i > t_i$, $\lim(t_i - s_i) = 0$ as $i \rightarrow \infty$ and that there should be no numbers t_1, \dots, t_{i-1} , s_1, \dots, s_{i-1} in the interval $[t_i, s_i]$. We shall define the sequence of vector functions $F^i(P)$ and of the real functions $\gamma_i(t)$ ($i = 1, 2, \dots$) with the following properties:

$$(12) \quad F^i(t, X) \quad (i = 1, 2, \dots) \quad \text{are functions continuous in } R^3$$

and of class C^∞ for $t \neq t_j$ ($j = 1, \dots, i$) $|F^i(t, X)| = 0$ for $|X| \geq i$.

$$(13) \quad d(P; F^i(-)) = 0 \quad (i = 1, 2, \dots).$$

(14) The functions $\gamma_i(t)$ ($i=1,2,\dots$) are continuous for any t and positive for $t \neq t_j$ ($j=1,\dots,i$).

(15) $\gamma_i(t) \leq 1/2 \gamma_{i-1}(t)$ ($i \geq 2$), $\gamma_1(t) \leq 1$.

(16) $|F^i(P) - F^{i-1}(P)| \leq 1/2 \gamma_{i-1}(t)$ ($i \geq 2$).

(17) For any vector functions $E(P)$ continuous in R^3 and satisfying the condition $|E(P) - F^i(P)| \leq \gamma_i(t)$ we have $d(P; E(-)) \leq 1/i$ for $|P| \leq i$ ($i=1,2,\dots$) and $c(s_i; t_i, X; E(-)) > 0$ for $|X| \leq i$.

In order to define the functions $F^i(P)$, $\gamma_i(t)$ it is sufficient to apply lemma 4 to $G(P) = 0$, $\delta = 1$, $\tau = t_1$, $\sigma = s_1$, $M = 1$, and to set $F^1(P) = K(P)$, $\gamma_1(t) = \min(\gamma(t), 1)$. Suppose that we have already defined the functions $F^k(P)$, $\gamma_k(t)$ in such a way that relations (12)–(17) are satisfied for $i = k$. In order to define the functions $F^{k+1}(P)$, $\gamma_{k+1}(t)$ we shall again apply lemma 4 to $G = F^k$, $\delta = \frac{1}{2} \min \gamma_k(t)$, $t \in [t_{k+1}, s_{k+1}]$, $\tau = t_{k+1}$, $\sigma = s_{k+1}$, $M = k+1$, setting $F^{k+1} = K$, $\gamma_{k+1}(t) = \min(\frac{1}{2} \gamma_k(t), \gamma(t))$.

It is easy to see that a sequence constructed in this way satisfies properties (12)–(17). By relations (14)–(16) the sequence of functions $F^i(P)$ is uniformly convergent to a certain function $F(P)$ as $i \rightarrow \infty$. $F(P)$ is continuous and bounded, and by (15), (16) it satisfies the inequality

(18) $|F(P) - F^i(P)| \leq \gamma_i(t)$ ($i=1,2,\dots$).

By (17) and (18) we have $d(P; F(-)) = 0$ in R^3 , i. e., each point of R^3 is a point of left-sided uniqueness for Eq. (1). We shall now show that each point of R^3 is a point of right-sided non-uniqueness for Eq. (1). Let $P^* = (t^*, X^*)$ be an arbitrary point of R^3 and ε any positive number. Let $X(t)$ be an integral satisfying the condition $X(t^*) = X^*$ defined for $t^* \leq t \leq t^* + 1$. Let us fix the index i in such a way that $|X(t)| \leq i$ in this interval and that $t^* < t_i < s_i < t^* + \min(1, \varepsilon)$. Hence, it follows by (17) and (18) that $c(s_i; t_i, X(t_i); F(-)) > 0$ and thus also $c(s_i; t^*, X^*, F(-)) > 0$.

We have thus completed the proof of the properties of Eq. (1) stated at the beginning of the paper.

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On the Multiplicative Linear Functionals on the Cartesian Product of Algebras

by

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Presented by S. MAZUR on March 18, 1957

It is well known that if R is an algebra with unit e over a field K , and if f is a multiplicative linear functional with values from K defined over R , then the set

$$(1) \quad M_f = \bigcap_{x \in R} f(x) = 0$$

is a maximal ideal in R and every element $z \in R$ is uniquely representable in the form

$$(2) \quad z = m + ae,$$

where $m \in M_f$, and $a = f(z) \in K$.

Now let R_t , $t \in T$, be a family of algebras over K , where T is an arbitrary abstract set; we assume that $\bar{K} \geq \aleph_0$. By $e(t)$ we denote the unit element of the algebra R_t . We define R as the cartesian product of the algebras R_t :

$$(3) \quad R = \prod_{t \in T} R_t;$$

its elements may be considered as functions defined over T which correlate with every $t \in T$ an element $x(t) \in R_t$ called the projection of x on R_t .

If f_{t_0} is a multiplicative linear functional defined on R_{t_0} , then the functional

$$(4) \quad (x) = f_{t_0}[x(t_0)]$$

is a multiplicative linear functional defined on R . In language of maximal ideals formula (4) may be written in the form

$$(5) \quad M_f = M_{f_{t_0}} \times \prod_{\substack{t \in T \\ t \neq t_0}} R_t,$$

where M_f , $M_{f_{t_0}}$ are the maximal ideals in R and in R_{t_0} .

The following question arises: is every multiplicative linear functional defined over R given by the formula (4)? The answer is given in the following

FUNDAMENTAL THEOREM. *If $\bar{K} < \aleph_I$ (by \aleph_I we denote the first aleph of measure $\neq 0$ *), then every multiplicative linear functional defined over R is given by the formula (4) if, and only if, every $(0-1)$ σ -measure defined over the field of all subsets of T is the atomic one, i. e., $\bar{T} < \aleph_I$ **).*

This theorem is obtained from two of the following three theorems which will be proved below:

THEOREM 1. *If $\bar{T} < \aleph_I$, then every multiplicative linear functional f is given by (4) (without here assuming that $\bar{K} < \aleph_I$).*

THEOREM 2. *If $\bar{T} \geq \aleph_I$ and $\bar{K} < \aleph_I$, then there exists a family R_t , $t \in T$, such that there exists a multiplicative linear functional f defined over R which cannot be written in the form (4).*

THEOREM 3. *If $\bar{K} \geq \bar{T}$, then every multiplicative linear functional defined on R may be written in the form (4).*

Theorems based upon a similar idea are known in literature [1]–[3].

Proof of Theorem 1. It is well known that if μ is a $(0-1)$ σ -measure, then the family \mathfrak{M} of subsets of T defined by the formula

$$(6) \quad \mathfrak{M} = \bigcup_{A \subset T} \mu(A) = 0$$

is a maximal ideal of subsets of T , i. e., it satisfies the following conditions:

- (α) If $A \subset T$, then either $A \in \mathfrak{M}$, or $T - A \in \mathfrak{M}$.
- (β) The family \mathfrak{M} is hereditary, i. e., if $A \subset B$ and $B \in \mathfrak{M}$, then $A \in \mathfrak{M}$.
- (γ) If $A_n \in \mathfrak{M}$, $n=1, 2, \dots$ then $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{M}$.

On the other hand, if \mathfrak{M} is any family satisfying α , β and γ , then there exists a $(0-1)$ σ -measure μ , such that \mathfrak{M} is given by the formula (6).

Now let f be a multiplicative linear functional defined on R . We put

$$(7) \quad \mathfrak{M}_f = \bigcup_{A \subset T} \chi_A \in \mathfrak{M}_f,$$

*) A cardinal m is said to be of measure $\neq 0$ if there exists a $(0-1)$ σ -measure defined on the field of all subsets of set of power m which vanishes on the points. A measure μ is called $(0-1)$ if it assumes only two values: 0 and 1.

**) In our first draft this theorem was proved in the case when K is the field of reals. The generalisation was suggested to the authors by Professor J. Łoś.

where χ_A is the characteristic function of the set A , i. e.,

$$(8) \quad \chi_A(t) = \begin{cases} e(t) & \text{for } t \in A, \\ 0 & \text{for } t \notin A. \end{cases}$$

We shall prove that \mathfrak{M}_f satisfies α , β and γ .

(ad α) If $f(\chi_A) = \alpha \neq 0$ and $f(\chi_{T-A}) = \beta \neq 0$, then

$$f(\beta\chi_A - \alpha\chi_{T-A}) = 0,$$

but this is impossible, for there exists the inverse element

$$\frac{1}{\beta}\chi_A - \frac{1}{\alpha}\chi_{T-A}.$$

(ad β) If $A \subset B$ and $\chi_B \in M_f$ then

$$\chi_A = \chi_A \cdot \chi_B \in M_f.$$

(ad γ) Let $\chi_{A_n} \in M_f$ and $A = \bigcup_{n=1}^{\infty} A_n$; we shall prove that $\chi_A \in M_f$.

We put $B_n = A_n - \bigcup_{k=1}^{n-1} A_k$; we have $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ and $B_k \cap B_s = \emptyset$ for $k \neq s$. We put

$$(9) \quad z(t) = \begin{cases} k_n \cdot e(t) & \text{for } t \in B_n, \\ 0 & \text{for } t \notin A, \end{cases}$$

where $k_n \in K$ for $n=1, 2, \dots$ and $k_i \neq k_l$ for $i \neq l$.

We shall prove that $z \in M_f$. Suppose that $z \notin M_f$, then $z = m + \alpha e$ and $\alpha \neq 0$. We have $u = z - \alpha e \in M_f$. If $\alpha \neq k_n$ for $n=1, 2, \dots$, then there exists an inverse

$$u^{-1}(t) = \begin{cases} \frac{1}{k_n - \alpha} e(t) & \text{for } t \in B_n, \\ -\frac{1}{\alpha} & \text{for } t \notin A, \end{cases}$$

which is impossible. If $\alpha = k_{n_0}$, then $v = u + \chi_{B_{n_0}} \in M_f$, but there exists an inverse

$$v^{-1}(t) = \begin{cases} \frac{1}{k_n - k_{n_0}} e(t) & \text{for } t \in B_n \quad n \neq n_0, \\ e(t) & \text{for } t \in B_{n_0}, \\ 1/k_{n_0} & \text{for } t \notin A, \end{cases}$$

which is also impossible; we have then $z \in M_f$, and

$$\chi_A = z \cdot x \in M_f,$$

where

$$x(t) = \begin{cases} \frac{1}{k_n} e(t) & \text{for } t \in B_n, \\ 0 & \text{for } t \notin A. \end{cases}$$

We now suppose that every $(0-1)$ σ -measure is an atomic one. Then every maximal ideal \mathfrak{M} may be written in the form

$$(10) \quad \mathfrak{M} = \bigcup_{A \in T} t_0 \bar{\epsilon} A,$$

where t_0 is an element of T . If f is a multiplicative linear functional on R , then we have

$$(11) \quad \mathfrak{M}_f = \bigcup_{A \in T} t_0 \bar{\epsilon} A,$$

and $\chi_{T-(t_0)} \in \mathfrak{M}_f$. From this follows at once that $\{0\} \times \prod_{t \neq t_0} R_t \subset \mathfrak{M}_f$ and (5) holds.

Proof of Theorem 2. We now suppose that on the set T there exists a non-atomic $(0-1)$ σ -measure μ and that $\bar{K} < \mathfrak{s}_T$. It is to be shown that there exists a family R_t , $t \in T$ of algebras, such that on the product $R = \prod_{t \in T} R_t$ there exists a multiplicative linear functional which cannot be written in form (4).

We put $R_t = K$ for every $t \in T$. Then R is the set of the all functions defined on T with values from K .

Let $x \in R$; we have $T = \bigcup_{p \in K} T_{x,p}$, where

$$(12) \quad T_{x,p} = \bigcup_{t \in T} x(t) = p,$$

We shall show that there exists such a p that $\mu(T_{x,p}) = 1$.

Let $K_0 = x(T)$. We put

$$(13) \quad \tilde{\mu}(A) = \mu[x^{-1}(A)];$$

it is obvious that $\tilde{\mu}$ is a $(0-1)$ σ -additive measure defined on K_0 . But $\bar{K}_0 \leq \bar{K} < \mathfrak{s}_T$ and hence $\tilde{\mu}$ must be an atomic measure and there exists a point $p \in K_0$ such that $\mu(p) = 1$. We have thus $\mu(T_{x,p}) = 1$ for $T_{x,p} = x^{-1}(p)$. Obviously, such a p is unique. We put

$$(14) \quad F(x) = p.$$

It is easy to see that F is a multiplicative linear functional which cannot be written in the form (4).

Proof of Theorem 3. Let $\bar{K} \geq \bar{T}$. Let M be a maximal ideal in R . There are two possibilities: either there exists a t_0 such that $M|_{R_{t_0}} \neq R_{t_0}$, or for every t we have $M|_t = R_t$. (By $A|_t$ we denote the set $\bigcup_{x \in A} x$.) In the first case M may be written in the form

$$(15) \quad M = M_{t_0} \times \prod_{t \neq t_0} R_t,$$

where M_{t_0} is a maximal ideal in R_{t_0} . If $M = M_f$ for an f in R , then f may be written in the form (4).

We shall now show that in the second case no functional corresponds to M . Suppose that $M_t = R_t$ for every $t \in T$ and that there exists a multiplicative linear functional f corresponding to this ideal. Then every $z \in R$ may be written in the form $z = m + ae$, where $m \in M$, and $a \in K$. We choose z in such a manner that $z(t) = k_t \cdot e(t)$, $k_t \in K$ and $k_t \neq k_{t'}$ for $t \neq t'$. This is possible because $\bar{K} \geq T$. Putting $u = z - ae$, we have $u \in M$ and $u(t) = (k_t - a)e(t)$. If $k_t \neq a$ for every t , then there exists an inverse

$$u^{-1}(t) = \frac{1}{k_t - a} e(t),$$

which is impossible. Thus we have $a = k_{t_0}$ for a $t_0 \in T$. But $M_{t_0} = R_{t_0}$ and there exists a $y \in M$ such that $y(t_0) = e(t_0)$. Then $\chi_{(t_0)} = \chi_{(t_0)} \cdot y \in M$. We put $v = u + \chi_{(t_0)}$ and have $v \in M$, but this is impossible for otherwise there would exist an inverse

$$v^{-1}(t) = \begin{cases} \frac{1}{k_t - a} e(t) & \text{for } t \neq t_0, \\ e(t_0) & \text{for } t = t_0, \end{cases}$$

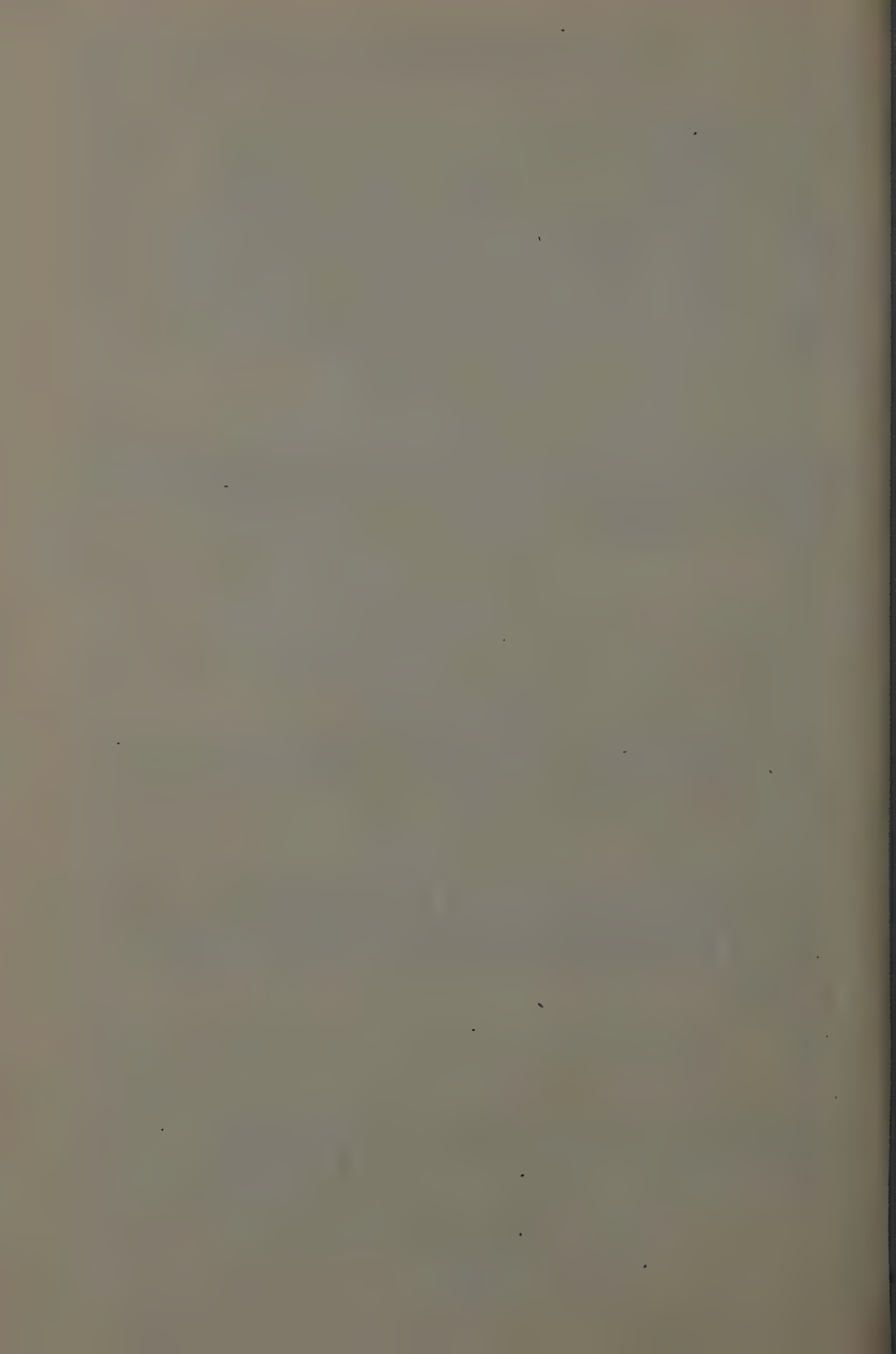
COROLLARY. If $T \geq s_I$, then the cartesian product of functionally continuous algebras is functionally continuous.

A topological algebra is called functionally continuous if every multiplicative linear functional defined on it is continuous.

This corollary follows at once from (5) and from the fact that if the set $M_{f_{t_0}}$ is closed, then so is M_f .

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On Chords of Convex Curves

by

H. STEINHAUS

Presented on March 23, 1957

THEOREM. *Let K be a convex curve. Three angles a, b, c with $a + b + c = 180^\circ$ being given, there exist three chords of K with a common midpoint, the angles between the consecutive halves being a, b, c, a, b, c .*

Before we proceed to prove our theorem by almost elementary means let us define some notions and explain some locutions employed in the sequel.

We call *convex* a plane, smooth, closed curve K which has with no line more than two points in common. *Smoothness* means that for any two lines R_1, R_2 coplanar with K , the product $R_1 R_2 K$ never reduces to a single point unless $R_1 \equiv R_2$. A *chord* of K is produced by a line R cutting K in P and P' ; its direction is R and its *midpoint* halves the segment PP' ; the distance PP' is the *length* of the chord, and the case $P = P'$ is not excluded; in this case we call the tangent in P the direction of the chord. \mathcal{R} is the open region interior to K .

Let OXY be a rectangular frame which enables us to employ such words as horizontal, vertical, right, left, above, beneath, etc., in the usual way. The axes OX, OY also give a meaning to the *positive turn* on K .

LEMMA 1. *The midpoints of all horizontal chords of K form a simple arc S joining continuously the lowest point P of K with the highest P' , through \mathcal{R} .*

LEMMA 2. *Let us draw through every point of the arc S defined in Lemma 1 a vertical chord: there exists a point on S which halves the corresponding chord.*

LEMMA 3. *There is only one point on S having the property stated in Lemma 2.*

To prove Lemma 1 take P as origin for the new co-ordinates u and v , the tangent to K in P as the v -axis, and PP' as the u -axis. Then the

arc PP' of K can be defined by a continuous function $v=f(u)$, and the arc $P'P$ by another continuous function $v=g(u)$; $v=\frac{1}{2}(f(u)+g(u))$ defines the arc S as the path of a point $Q(u)$.

To prove Lemma 2 let us call $h(u)$ the length of the part of the vertical chord through $Q(u)$, which lies below $Q(u)$; we immediately obtain $h(0)=0$, $h(s)$ = length of the chord PP' for s = length of same chord. For the length of the upper part $k(u)$ we obtain $k(0)=s$, $k(s)=0$. Both functions being continuous there is a u_0 ($0 < u_0 < s$) for which $h(u_0)=k(u_0)$ *q. e. d.*

To prove Lemma 3 indirectly let us admit that there are two different points, Q and Q' , each of them halving both the horizontal and the vertical chords passing through them. Q and Q' are consequently the respective centres of two crosses with all 8 vertices on K . Fig. 1 shows

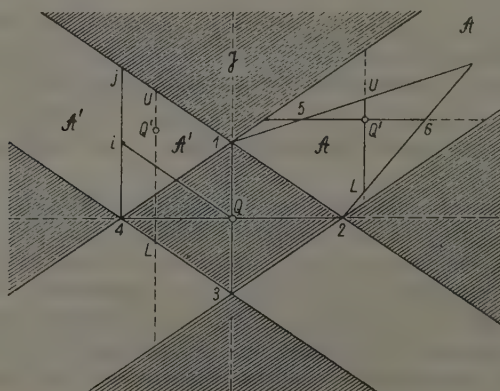


Fig. 1

the rhombus formed by the vertices 1234 of the first cross; its sides divide the whole plane into 9 regions, alternatively white and shaded. No vertex of the second cross can lie in the shaded regions: the interior of the rhombus being a part of \mathcal{R} , it cannot contain such vertices; as to the infinite shaded region \mathcal{J} , a vertex lying in it would form with 2 and 4 a triangle containing 1 in its interior which is obviously a part of \mathcal{R} , against 1 lying on K . The same applies to other infinite shaded regions. Consequently, all vertices of the second cross are confined to the white regions and their frontiers. Let us suppose the left vertex 5 and the right vertex 6 of the second cross to lie both in \mathcal{A} . The lower vertex cannot lie between Q' and L because it is not allowed to lie in the interior of the quadrangle 1562; it must therefore have its place in L or beneath L . The upper vertex cannot lie above U , as it would form with 1 and 6 a triangle enclosing 5. As $Q'U$ is shorter than $Q'L$,

the two last statements are contradictory. The only possibility left is a cross congruent to the first with its vertices on the frontiers of \mathcal{A} , but it leads to three different collinear vertices at least, against our definition of convexity. Let us now try to place 5 in \mathcal{A}' and 6 in \mathcal{A} ; because of symmetry we are allowed to assume Q' to lie on the left half of our sketch. Such assumptions do confine Q' to the interior of $ijlQ$ or to the sides of this parallelogram. The upper vertex must lie beneath U . It forms with 4 and 1 a triangle and the lower vertex must keep outside, which compels it to lie beneath L ; as we have again $Q'U < Q'L$, both conditions are contradictory; the admission of frontiers leads again to collinearity. The line 24 being an axis of symmetry, no other solutions are left.

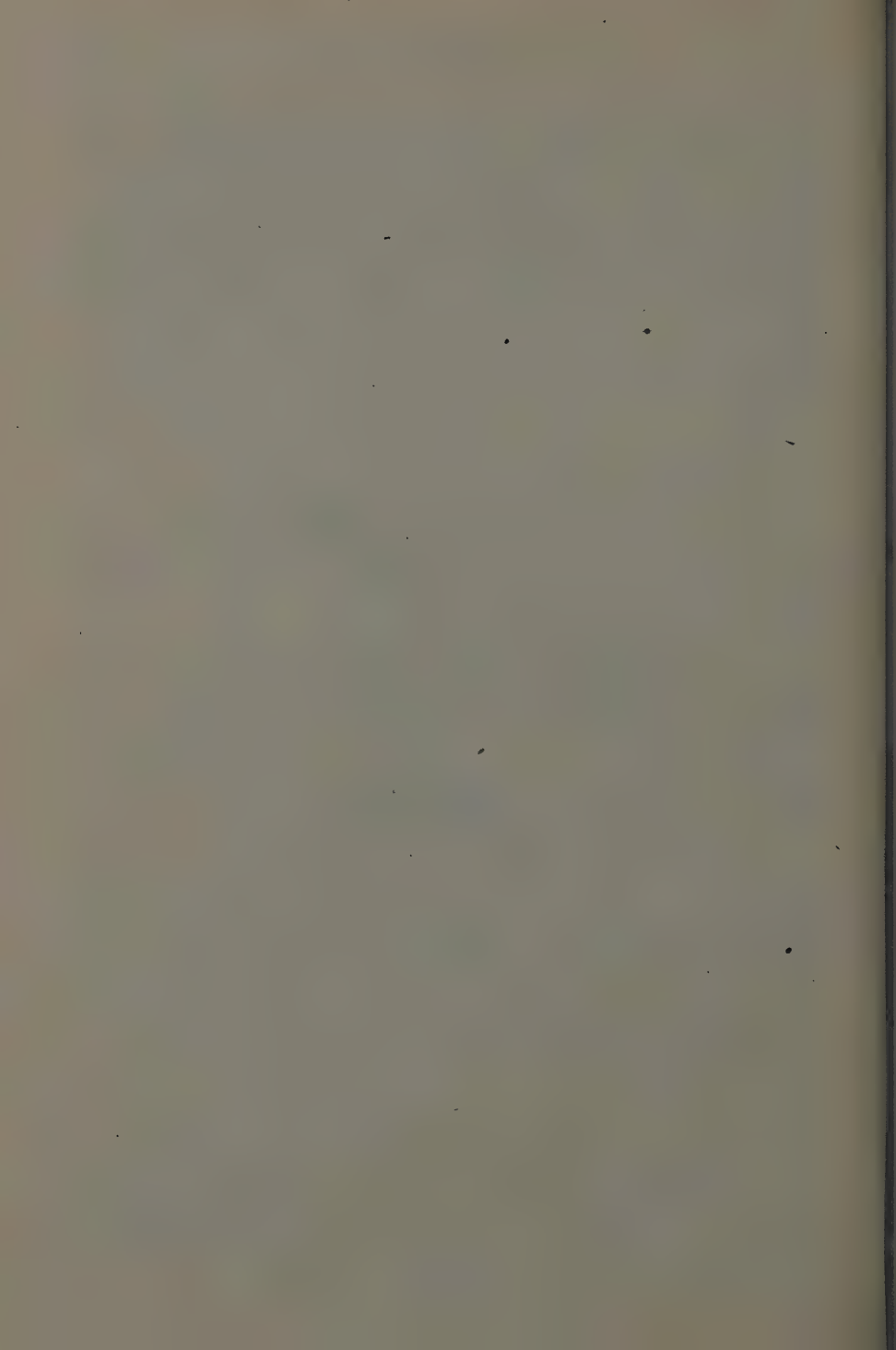
LEMMA 4. *Lemmas 2 and 3 remain valid when replacing the vertical chords by a family of parallel chords inclined by an arbitrary angle a ($0 < a < 180^\circ$) against the horizontal direction.*

To prove this lemma let us apply an affine transformation conserving the horizontal direction and changing the family specified in the lemma into a vertical family. As convexity, continuity, parallelism and halving are affine invariants, the application of Lemmas 2 and 3 to the changed curve implies Lemma 4 for K .

It results immediately from Lemmas 2, 3 and 4 that for any pair of different directions there exists one and only one pair of chords having these directions and a common midpoint. Let a be the angle between the chords and Q their midpoint. Keeping a constant and turning both directions through an angle t we obtain $Q(t)$ as the new midpoint. $Q(t)$ is a well-defined function of t ; this circumstance and the obvious closedness of the set of all points $Q(t)$ shows the continuity of $Q(t)$.

The last step of the proof consists in drawing through $Q(t)$ a third chord inclined by the angle b to the second one ($0 < a < a + b < 180^\circ$). $Q(t)$ divides this chord into two parts of lengths $T_1(t)$ and $T_2(t)$ respectively. When t increases from 0 to 180° , both parts move continuously and they interchange their positions, which implies $T_1(\tau) = T_2(\tau)$ for a certain τ . Thus $Q(\tau)$ halves three chords, the angles between them being a, b, c .

The question as to whether our theorem applies to every plane closed curve remains open.



On (S)- and (DS)-Spaces

by

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Presented by S. MAZUR on March 29, 1957

Introduction. The notion of (S)-space was introduced in [1]. The strong and the strongest dual of an (S)-space are identical and will be named (DS)-spaces. In this paper is given a characterisation of (DS)-spaces.

Definition. A locally convex space X is named (M)-space [1] if every bounded subset of X is totally bounded (precompact); X is named (S)-space if to every continuous pseudonorm $|x|$ defined on X there is a neighbourhood of zero which is totally bounded with respect to $|x|$ [1].

It is evident that every (S)-space is an (M)-space.

Definition. Let (x_ξ) be a certain MS-sequence (Moore-Smith sequence) defined in a locally convex linear space X ; (x_ξ) is named *bounded* if it is bounded with respect to each continuous pseudonorm on X , i. e., if for every continuous pseudonorm $|x|$ there is such a ξ_0 that the set $\{(x_\xi): \xi \geq \xi_0\}$ is bounded. The element $x_0 \in X$ is called a *cluster point* of (x_ξ) if $x_0 \in \bigcap_{\xi} Z_\xi$, where Z_ξ is the closure of $\{x_\eta: \eta \geq \xi\}$.

THEOREM 1. *If the space X is a complete (S)-space, then every bounded MS-sequence has at least one cluster point.*

Definition. A norm $\|x\|$ defined on a linear subset Z of a locally convex space X is named *restrictive* on Z if each continuous pseudonorm defined on X is continuous on Z with respect to $\|x\|$; the space X is named (DS)-space if to every bounded subset $B \subset X$ there is a norm $\|x\|_Z$ which is restrictive on $Z \supset B$ and such that B is totally bounded with respect to $\|x\|_Z$.

THEOREM 2. *Every (DS)-space is an (M)-space.*

THEOREM 3. a. *If X is a bornological or tonnelé (S)-space, then the strongest dual of X is identical with the strong dual of X and it is a (DS)-space.*

b. If X is a (DS) -space, then the strong dual of X is an (S) -space.

c. The strong dual and the strongest dual of a metrisable (M) -space are identical.

Let us consider the (B_0) -space R_0 of all sequences $x = (x_{m,n})$ such that

$$\sup_{m,n} m^k a_{m,n}^l |x_{m,n}| < +\infty$$

for $k, l = 1, 2, \dots$, where

$$a_{m,n}^{(l)} = \begin{cases} n^{l-m} & \text{for } m \leq l \\ 1 & \text{for } m > l \end{cases}$$

with the norms

$$|x|_{k,l} \stackrel{\text{def}}{=} \sup_{m,n} m^k a_{m,n}^{(l)} |x_{m,n}|.$$

The space R_0 is an (M) -space which is not any (S) -space.

THEOREM 4. Every metrisable (M) -space is a (DS) -space.

COROLLARY. Every (DB_0) -space, which is also an (M) -space is an (S) -space.

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An Approximate Method for Solving Linear Equations in a Hilbert Space

by

M. ALTMAN

Presented by W. ORLICZ on March 30, 1957

1. We have applied the results of paper [1] in paper [2] to the Gaussian principle of least squares in the error theory. Thus we have obtained an iterative method for the approximate solution of error equations by the Gaussian principle. In the present note we give an iterative method for the approximate solution of linear equations in Hilbert space. This method is also based on the results of paper [1] and is closely connected with the well-known method of least squares in Hilbert space. The application of the latter method consists in finding the orthogonal projections on the finite-dimensional subspaces. In contradistinction to the method of least squares, using the iterative method given here, we find the approximate orthogonal projections on the finite-dimensional subspaces. These approximate projections can be found also by the Gauss-Seidel iterative method. In this case we obtain some formulae for the error estimate, which also follow from the results of paper [1].

Let H be a Hilbert space and denote by $H_n \subset H$ the finite-dimensional subspace of H spanned by n linearly independent vectors a_1, a_2, \dots, a_n . We shall consider the approximate value of the orthogonal projection onto the subspace H_n . For this purpose we apply the results of paper [1].

Let l be an arbitrary vector of H such that $l \notin H$. Denote by P_i the orthogonal projection onto the one-dimensional subspace generated by the vector a_i .

For the vector l we define the infinite sequence of vectors y_p as follows:

$$(1) \quad \begin{aligned} y_1 &= l - P_1 l, & y_2 &= y_1 - P_2 y_1, \dots, & y_n &= y_{n-1} - P_n y_{n-1}, \\ y_{n+1} &= y_n - P_1 y_n, \dots, & y_{nk+i} &= y_{nk+i-1} - P_i y_{nk+i-1}, \\ & & & \text{for } i=1, 2, \dots, n; & k &= 1, 2, \dots \end{aligned}$$

Hence, we have

$$(2) \quad y_1 = l - \|a_1\|^{-2}(l, a_1)a_1, \dots, y_{nk+i} = y_{nk+i-1} - \|a_i\|^{-2}(y_{nk+i-1}, a_i)a_i \\ \text{for } i = 1, 2, \dots, n; \quad k = 0, 1, 2, \dots, \quad \text{where } y_0 = l.$$

Exactly as in paper [1] it may be proved that

$$(3) \quad y_p \rightarrow l - Q_n l \quad \text{as } p \rightarrow \infty,$$

where $Q_n l$ is the orthogonal projection of l into H_n . If $Q_n l = \sum_{i=1}^n t_i a_i$, then

$t_i = \sum_{k=0}^{\infty} \|a_i\|^{-2} (y_{nk+i-1}, a_i)$. Thus $Q_n l = \sum_{i=1}^n \sum_{k=0}^{\infty} \|a_i\|^{-2} (y_{nk+i-1}, a_i) a_i$ and the approximate orthogonal projection $\tilde{Q}_n l$ of l onto H_n is given by the formula

$$(4) \quad \tilde{Q}_n l = \sum_{i=1}^n \sum_{k=0}^{k_n} \|a_i\|^{-2} (y_{nk+i-1}, a_i) a_i,$$

where k_n is appropriately chosen.

It follows from (1) that

$$(5) \quad \sum_{i=1}^n \sum_{k=0}^p \|a_i\|^{-2} (y_{nk+i-1}, a_i) a_i = l - y_{n(p+1)}.$$

Consider now the linear equation

$$(6) \quad Ax = l \quad (x, l \in H),$$

where A is a linear (i. e., additive and homogeneous) operator with domain $D(A) \subset H$ and range in H .

Let $(e_i) \ i=1, 2, \dots$ be a linearly independent system in H and put $a_i = Ae_i$ ($i=1, 2, \dots$).

Let us assume that the linear operator A satisfies the following conditions:

1° there exists a positive number c such that

$$(7) \quad \|Ax\| \geq c\|x\| \quad \text{for every } x \text{ of } D(A);$$

2° there exists a solution of Eq. (6);

3° the closed linear subspace of H spanned by the elements a_1, a_2, \dots contains the element l .

We now define the iterative method for the solution of Eq. (6). The approximate solutions x_n are obtained by repeating the construction of the sequence (1) for every H_n ($n=2, 3, \dots$) as follows:

$$x_1 = \|a_1\|^{-2}(l, a_1)e_1, \quad \text{hence} \quad Ax_1 = l - y_1.$$

If for a certain H_n there are defined the sequence (y_p) according to (1), and the approximate orthogonal projection according to (4), then

for the approximate solution x_n we put

$$(8) \quad x_n = \sum_{i=1}^n \sum_{k=0}^{k_n} \|a_i\|^{-2} (y_{nk+i-1}, a_i) e_i.$$

Hence, we have by (5)

$$(9) \quad \tilde{Q}_n l = A x_n = l - y_{p_n}.$$

To obtain the approximate solution x_{n+1} we repeat the construction of the sequence (1) for H_{n+1} , where instead of l we take y_{p_n} , i. e., $y_{p_{n+1}} = y_{p_n} - P_1 y_{p_n}, \dots$ Further, we find the approximate orthogonal projection $\tilde{Q}_{n+1} l$ of l into H_{n+1} by the formula

$$(10) \quad \tilde{Q}_{n+1} l = \tilde{Q}_n l + \sum_{i=1}^{n+1} \sum_{k=0}^{k_{n+1}} \|a_i\|^{-2} (y_{p_n+(n+1)k+i-1}, a_i) a_i.$$

Hence, $x_{n+1} = x_n + \sum_{i=1}^{n+1} \sum_{k=0}^{k_{n+1}} \|a_i\|^{-2} (y_{p_n+(n+1)k+i-1}, a_i) a_i$ and $A x_{n+1} = l - y_{p_{n+1}}$. For the error estimate we use the following evident formula:

$$(11) \quad \|y_{q+1}\|^2 = \|y_q\|^2 - \|a_{i_q}\|^{-2} (y_q, a_{i_q})^2.$$

If k_n in (4) and (8) is so chosen that $\|\tilde{Q}_n l - Q_n l\| < \varepsilon_n$, where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, then $\tilde{Q}_n l \rightarrow l$ as $n \rightarrow \infty$, since $Q_n l \rightarrow l$ as $n \rightarrow \infty$ by 3°. It follows from (9) that $y_p \rightarrow 0$ as $p \rightarrow \infty$. Thus we have by (7) and 2° that the sequence of approximate solutions x_n converges to a solution x of Eq. (6).

It is easy to see that $y_{p_n} - Q_{n+1} y_{p_n} = l - Q_{n+1} l$; hence, passing from H_n to H_{n+1} , we can replace l by y_{p_n} to obtain the approximate projection (10).

Remark. The choice of k_n in (4), corresponding to ε_n and satisfying the condition $\|\tilde{Q}_n l - Q_n l\| < \varepsilon_n$, is effectively very difficult (cf. [1]). But in practice this is not necessary. We carry out the computations of the elements of sequence (y_p) corresponding to H_n and $\tilde{Q}_n l$ until the sequence of numbers $\|y_p\|^2$, monotonously decreasing by (11), begins to diminish too slowly. We then pass from H_n to H_{n+1} or H_{n+m} and repeat the process. This is possible, since the error estimate for the approximate solution is given by (9).

It is convenient to use the following simple scheme of computations. If, for instance, $y_{p+1} = y_p - \|a_i\|^{-2} (y_p, a_i) a_i$, then the approximate co-ordinate \tilde{t}_i in (8) is improved by addition of $\|a_i\|^{-2} (y_p, a_i)$.

2. To find the approximate solution $x_n = \sum_{i=1}^n t_i^n e_i$ of Eq. (6) by the method of least squares we have the following system of linear algebraic equations:

$$(12) \quad \sum_{i=1}^n (A e_i, A e_k) t_i^n = (l, A e_k) \quad (k = 1, 2, \dots, n).$$

It is shown in paper [1] that the iterative method for finding the orthogonal projection on H_n defined by (1) coincides with the Gauss-Seidel process applied to system (12). Hence, we infer that the method given in section 1. coincides with the application of the Gauss-Seidel process to system (12) for $n=2,3,\dots$

Using the results of paper [1] we obtain a recurrent formula for the error estimate as follows.

If t_{ni}^k is the k -th approximate value of the i -th co-ordinate ($i=1,2,\dots, n$) defined by the Gauss-Seidel process applied to system (12) and $y_{p_n}^k =$

$= l - Ax_n^k$, where $x_n^k = \sum_{i=1}^n t_{ni}^k e_i$ is the approximate solution of Eq. (6),

then we have $y_{p_n+k}^k = l - Ax_n^{k+1}$, where $x_n^{k+1} = \sum_{i=1}^n t_{ni}^{k+1} e_i$, and

$$(13) \quad \|y_{p_n+k}^k\|^2 = \|y_{p_n}^k\|^2 - \sum_{i=1}^n (t_i^{k+1} - t_i^k)^2 \|a_i\|^2.$$

As in remark in section 1. if the monotonously decreasing sequence of numbers $\|y_p\|^2$ begins to diminish too slowly, we pass to system (12), where instead of n we take $n+1$ or $n+r$ and put zero for the initial approximate value of the co-ordinate t_{n+1} or t_{n+j} ($j=1,2,\dots,r$).

Notice that formula (13) results from (11) and from the equivalence of the methods given in sections 1. and 2.

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On the Approximate Solutions of Operator Equations in Hilbert Space

by

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Presented by W. ORLICZ on March 30, 1957

The present note contains an application of the method given in papers [1] and [2] to approximate solutions of non-linear and linear operator equations in Hilbert space.

Let H be a real Hilbert space and let $S(x_0, r)$ be a closed sphere in H with centre x_0 and radius r .

1. Consider the non-linear operator equation

$$(1) \quad P(x) = 0,$$

where P is a non-linear continuous operator defined on the sphere $S(x_0, r)$ with values in H . Let us assume that $P(x)$ is differentiable in the sense of Fréchet in the sphere $S(x_0, r)$ and denote by $P'(x)$ the Fréchet differential of $P(x)$.

Putting $F(x) = \|P(x)\|^2$ we reduce Eq. (1) to

$$(2) \quad F(x) = 0,$$

and applying the method of paper [2] to Eq. (2) we obtain an approximate process for the solution of Eq. (1), as in paper [2]:

$$(3) \quad \begin{aligned} x_1 &= x_0 - \frac{\|P(x_0)\|^2}{2\|Q(x_0)\|^2} Q(x_0), \\ x_{n+1} &= x_n - \frac{\|P(x_n)\|^2}{2\|Q(x_n)\|^2} Q(x_n), \end{aligned}$$

where $Q(x) = \bar{P}'(x) P(x)$, and the linear operator $\bar{P}'(x)$ is the adjoint of $P'(x)$.

Suppose that there exists the Fréchet differential $Q'(x)$ in the sphere $S(x_0, r)$ and that $Q'(x)$ is bounded, i. e., that there exists a constant K such that $\|Q'(x)\| \leq K$ for every x of $S(x_0, r)$.

Suppose also that the Fréchet differential $P'(x)$ possesses a continuous inverse which is bounded in $S(x_0, r)$, i. e., that there exists a constant B such that $\|P'(x)y\| \geq B^{-1}\|y\|$ for every y of H and every x of $S(x_0, r)$, and the range of the linear operator $P'(x)$ is necessarily the whole space H .

An existence theorem for the solution of Eq. (1) as well as a convergence theorem of process (3) to the solution of Eq. (1) are given by

THEOREM 1. *Let us assume that the following conditions are satisfied:*

1° *there exist two constants B and K such that*

$$(4) \quad B^2 K < 4;$$

2° *the Fréchet differential $P'(x)$ satisfies condition*

$$(5) \quad \|P'(x)y\| \geq B^{-1}\|y\| \quad \text{for every } y \text{ of } H \text{ and every } x \text{ of } S(x_0, r);$$

3° *the Fréchet differential $Q'(x)$ exists and satisfies the condition*

$$(6) \quad \|Q'(x)\| \leq K \quad \text{for every } x \text{ of } S(x_0, r);$$

4° *the error estimate for the first approximate solution (3) is known:*

$$(7) \quad \|x_1 - x_0\| = \frac{\|P(x_0)\|^2}{2\|Q(x_0)\|} < \eta_0$$

and

$$r = \frac{2\eta_0}{2 - B\sqrt{K}}.$$

Then Eq. (1) has a solution x^* which belongs to sphere $S(x_0, r)$ and the sequence of the approximate solutions x_n defined by process (3) converges to x^* . For the error estimate we have the formula

$$(8) \quad \|x_n - x^*\| \leq \eta_0 \frac{\alpha^n}{1 - \alpha},$$

where $\alpha = \frac{1}{2}B\sqrt{K}$.

Proof. It follows immediately from (3) that

$$(9) \quad \|P(x_n)\|^2 = (x_n - x_{n+1}, 2\bar{P}'(x_n)P(x_n)).$$

Hence, we have the identity *)

$$(10) \quad \|P(x_n)\|^2 = \|P(x_n)\|^2 - \|P(x_{n-1})\|^2 - (x_{n-1} - x_n, 2\bar{P}'(x_{n-1})P(x_{n-1})).$$

Using the analogue of Taylor's formula we have by (10) and (6)

$$(11) \quad \|P(x_n)\|^2 \geq K \|x_n - x_{n-1}\|^2.$$

*) If $Q(x_n)$ vanishes, then $P(x_n)$ also vanishes by (5); hence x_n is a solution of Eq. (1).

Thus, we obtain by (3), (5) and (11)

$$\|x_{n+1} - x_n\| = \frac{\|P(x_n)\|^2}{2\|\bar{P}'(x_n)P(x_n)\|} \leq \frac{B\|P(x_n)\|^2}{2\|P(x_n)\|} \leq \frac{B}{2}\sqrt{K}\|x_n - x_{n-1}\|.$$

Hence, we have

$$(12) \quad \|x_{n+1} - x_n\| \leq \frac{1}{2}B\sqrt{K}\|x_n - x_{n-1}\|.$$

All assertions of the theorem result from (12) and (7) in the usual way. Since $\|P(x_n)\| \rightarrow 0$ as $n \rightarrow \infty$ by (11), we infer that the limit point of sequence (3) is a solution of Eq. (1).

Remark 1. On account of (5), condition (7) can be replaced by the condition

$$(13) \quad \|x_1 - x_0\| \leq \frac{B}{2}\|P(x_0)\| \leq \eta_0.$$

Suppose now that the second Fréchet differential $P''(x)$ of $P(x)$ exists and is bounded in the sphere $S(x_0, r)$, i. e., that there exists a constant K such that

$$(14) \quad \|P''(x)\| \leq K \quad \text{for every } x \text{ of } S(x_0, r).$$

Modifying condition (6) we obtain the following theorem as a corollary of Theorem 1.

THEOREM 2. Let us assume that the following conditions are fulfilled:

1° there exists a constant D such that

$$(15) \quad \|P(x)\| \leq D \quad \text{for every } x \text{ of } S(x_0, r);$$

2° there exists a constant A such that

$$(16) \quad \|P'(x)\| \leq A \quad \text{for every } x \text{ of } S(x_0, r);$$

3° the Fréchet differential $P'(x)$ satisfies condition (5);

4° the second Fréchet differential $P''(x)$ satisfies condition (14);

5° the constants A, B, D and K are subject to the condition

$$B^2(A^2 + KD) < 4;$$

6° condition (7) is satisfied, and

$$r = \frac{2\eta_0}{2 - B\sqrt{A^2 + KD}} < \eta_0.$$

Then Eq. (1) has a solution x^* in the sphere $S(x_0, r)$ and the sequence of the approximate solutions x_n converges to x^* . For the error estimate we have the formula (8) putting in (8) $\alpha = \frac{1}{2}B\sqrt{A^2 + KD}$.

Remark 2. Using (5) and (15) we can replace condition 5° by the condition

$$\|x_1 - x_0\| \leq DB/2.$$

2. Now consider the linear equation

$$(17) \quad L(x) = y \quad x, y \in H,$$

where L is a linear bounded transformation of H on the whole of H , having an inverse L^{-1} .

Applying process (3) to this case we obtain the following sequence of approximate solutions x_n :

$$(18) \quad \begin{aligned} x_1 &= x - \frac{\|L(x) - y\|^2}{2\|\bar{L}(L(x) - y)\|^2} \bar{L}(L(x) - y), \\ x_{n+1} &= x_n - \frac{\|L(x_n) - y\|^2}{2\|\bar{L}(L(x_n) - y)\|^2} \bar{L}(L(x_n) - y), \end{aligned}$$

where the operator \bar{L} denotes the adjoint of L .

The following theorem is a particular case of Theorem 1:

THEOREM 3. Suppose that the linear bounded operator L defines a one-to-one mapping of H onto H satisfying the condition

$$(19) \quad \|L\| \cdot \|L^{-1}\| \leq 2\alpha < 2.$$

Then the sequence of approximate solutions x_n defined by process (18) converges to the solution of Eq. (17) for the arbitrary initial approach x of H . For the error estimate we have the following formula:

$$(20) \quad \|x_n - x^*\| \leq \frac{\|L(x) - y\|^2}{2\|\bar{L}(L(x) - y)\|^2} \cdot \frac{\alpha^n}{1 - \alpha} \leq \frac{\|L^{-1}\| \|L(x) - y\|}{2} \cdot \frac{\alpha^n}{1 - \alpha},$$

where α is defined by (19) and x^* is the solution of Eq. (17).

Notice that Theorem 3 can also be obtained independently of Theorem 1. For this purpose it is sufficient to observe that by formula (18) we have

$$(21) \quad (x_{n+1} - x_n, 2\bar{L}(L(x_n) - y)) = -\|L(x_n) - y\|^2.$$

Hence,

$$(22) \quad \|L(x_n) - y\|^2 = \|L(x_n) - y\|^2 - \|L(x_{n-1}) - y\|^2 - (x_n - x_{n-1}, 2\bar{L}(L(x_{n-1}) - y)).$$

But it is easy to see that for arbitrary x and Δx of H we have the identity:

$$(23) \quad \|L(x + \Delta x) - y\|^2 - \|L(x) - y\|^2 - (\Delta x, 2\bar{L}(L(x) - y)) = \|L(\Delta x)\|^2.$$

It follows from (18), (22), (23) and (19) that

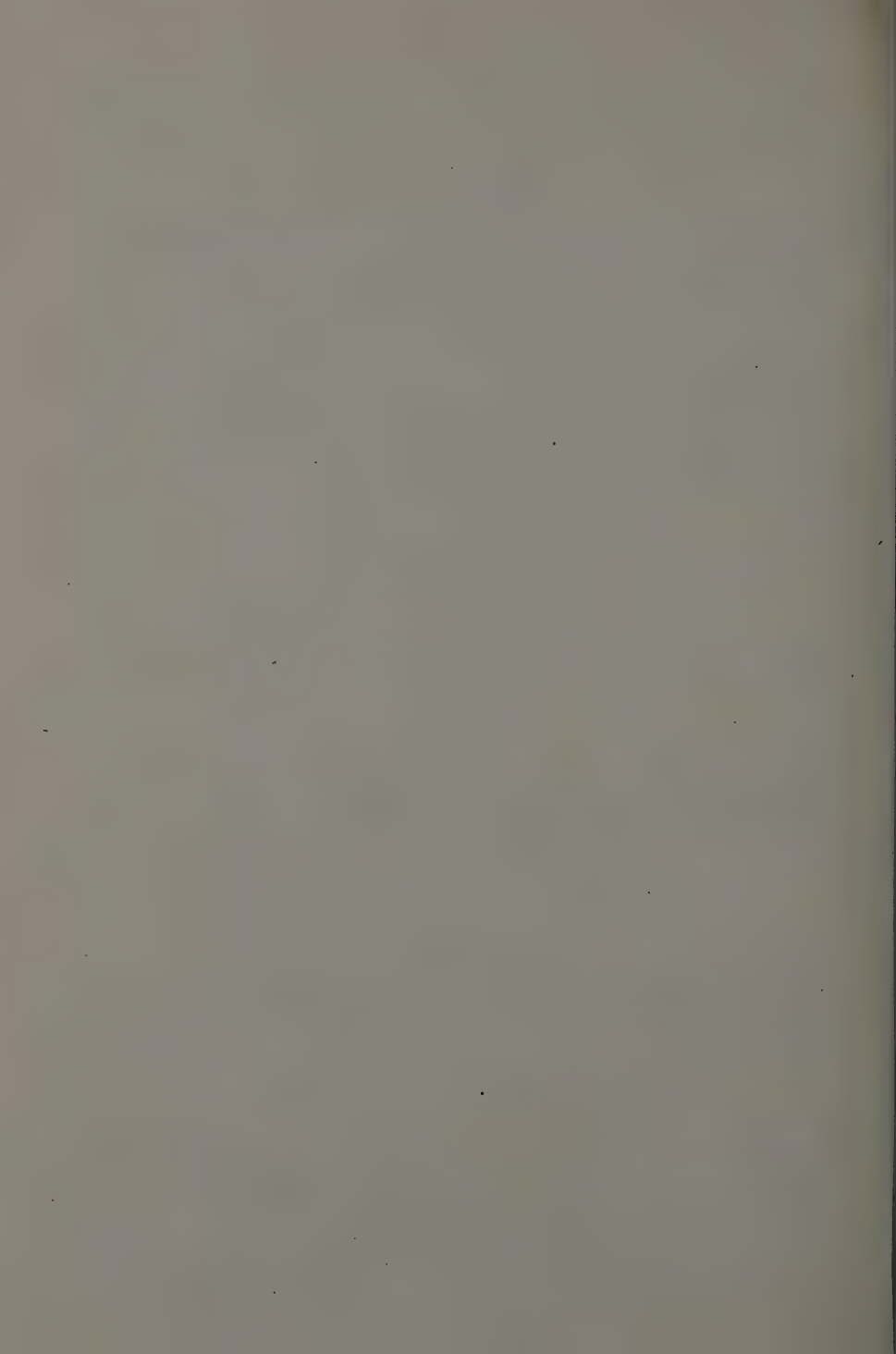
$$(24) \quad \|x_{n+1} - x_n\| \leq \frac{\|L^{-1}\| \|L(x_n) - y\|}{2} \leq \frac{\|L^{-1}\| \|L\|}{2} \|x_n - x_{n-1}\|.$$

The convergence of the sequence (x_n) to a certain element x^* of H results from (24). It follows from (21) that x^* is the solution of Eq.(17).

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On a Class of Saks Spaces

by

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Presented by W. ORLICZ on April 1, 1957

1. In this note we denote by $\varphi(u)$ a function convex and continuous for $u \geq 0$ satisfying the following conditions: a) $\varphi(0)=0$, $\varphi(u)>0$ for $u>0$; b) $\varphi(u)/u \rightarrow 0$ for $u \rightarrow 0$; c) $\varphi(u)/u \rightarrow \infty$ for $u \rightarrow \infty$. Further we denote by $\psi(u)$ the function complementary to $\varphi(u)$ [1]; $\psi(u)$ is a convex function (for $u \geq 0$) and satisfies the above conditions a)—c) as well. In the sequel $h_A(t)$ will denote the characteristic function of the set A . Moreover, if $x(t)$ is a real-valued measurable function in $(-\infty, \infty)$ then we shall write:

$$\mathfrak{I}_\varphi(x) = \int_{-\infty}^{\infty} \varphi(|x(t)|) dt, \quad \mathfrak{I}_\psi(x) = \int_{-\infty}^{\infty} \psi(|x(t)|) dt, \quad \mathfrak{I}(x) = \int_{-\infty}^{\infty} x(t) dt.$$

By L^{φ} or M we denote the class of all functions measurable in $(-\infty, \infty)$ such that for a certain $k>0$, $\mathfrak{I}_\varphi(kx)<\infty$ or all measurable essentially bounded functions in $(-\infty, \infty)$, respectively. When $\varphi(u)=u^\alpha$, where $\alpha \geq 1$, we write L^α instead of L^{φ} . The addition and scalar multiplication being defined as usual, classes L^{φ} and M are linear. Write $M^{\varphi} = L^{\varphi} \cap M$ and let M^φ be the class of functions belonging to the class M such that $\mathfrak{I}_\varphi(x)<\infty$. At first we remark:

1.1. THEOREM 1. *The inclusion $M^{\varphi_2} \subset M^{\varphi_1}$ holds if, and only if,*

$$(1) \quad \varphi_1(u) = O[\varphi_2(u)] \quad \text{for} \quad u \rightarrow 0.$$

Suppose (1), i. e. $\varphi_1(u) \leq c\varphi_2(u)$ for $0 \leq u < u_0$. Let $A = \{t: |x(t)| \geq u_0\}$, $B = (-\infty, \infty) - A$. Since $\mathfrak{I}_{\varphi_2}(x) \geq |A| \varphi_2(u_0)$, we have $|A| < \infty$ for $x \in M^{\varphi_2}$ and $\mathfrak{I}_{\varphi_1}(h_A x) \leq |A| \varphi_1(\sup^* |x(t)|)$ *, $\mathfrak{I}_{\varphi_1}(h_B x) \leq c \mathfrak{I}_{\varphi_2}(h_B x)$ imply $\mathfrak{I}_{\varphi_1}(x) < \infty$ if $\mathfrak{I}_{\varphi_2}(x) < \infty$. Now, let us suppose that (1) is not satisfied. There exists a sequence $u_n \downarrow 0$ such that $\varphi_1(u_n) \geq 2^n \varphi_2(u_n)$ for $n=1, 2, \dots$. Let us write

) $\sup^ y(t)$ will denote the essentially supremum of the function $y(t)$ in $(-\infty, \infty)$.

$(-\infty, \infty) = \bigcup_1^\infty \delta_n$, where δ_n are disjoint intervals such that $|\delta_n|^{-1} = \varphi_2(u_n) 2^n$.

Define a function $x = x(t)$ by the formula $x(t) = u_n$ for $t \in \delta_n$. Then $\mathfrak{I}_{\varphi_1}(x) = \infty$, $\mathfrak{I}_{\varphi_2}(x) = 1$.

1.2. COROLLARY. The above result implies immediately:

The equality $M^\varphi = M^{\varphi^}$ holds if, and only if, the following condition is satisfied:*

$$\varphi(2u) = O[\varphi(u)] \quad \text{for } u \rightarrow 0 \quad (\text{condition } (\Delta_2) \text{ for small } u).$$

We remark that M^{φ^*} is always a linear class.

1.3. We introduce in M^{φ^*} norms

$$(2) \quad \|x\| = \sup^* |x(t)|;$$

$$(3) \quad \|x\|^* = \inf l,$$

where the infimum is to be taken over the set of all numbers $l > 0$ such that $\mathfrak{I}_\varphi(x/l) \leq 1$;

$$(4) \quad \|x\|^\circ = \sup |\mathfrak{I}(xy)|,$$

where the supremum is extended over the set of functions y such that $\mathfrak{I}_\varphi(y) \leq 1$. We remark that formulae (3) and (4) define norms also in the whole L^{φ^*} . As is well known, these norms are homogenous and satisfy the inequalities $\|x\|^* \leq \|x\|^\circ \leq 2\|x\|^*$ for $x \in L^{\varphi^*}$. The space L^{φ^*} is complete with respect to the norm $\|\cdot\|^*$ (or $\|\cdot\|^\circ$). Let $M_s^{\varphi^*} = \{x : x \in M^{\varphi^*}, \|x\| \leq 1\}$; if we define in $M_s^{\varphi^*}$ the metric by the formula $d(x, y) = \|x - y\|^*$ then $M_s^{\varphi^*}$ becomes a Saks space (for the definition of Saks space, see e. g. [3], [4]).

1.4. THEOREM 2. *The relations*

$$(a) \quad \|x_n\|^* \rightarrow 0 \text{ for } n \rightarrow \infty, x_n \in M_s^{\varphi^*}$$

and

$$(b) \quad \mathfrak{I}_\varphi(x_n) \rightarrow 0 \text{ for } n \rightarrow \infty, x_n \in M_s^{\varphi^*}$$

are equivalent if, and only if, the condition (Δ_2) for small u is satisfied.

Sufficiency. The condition (Δ_2) for small u implies the existence of constants $c > 0$, $u_0 > 0$ such that the inequality $\varphi(2u) \leq c\varphi(u)$ for $0 \leq u \leq u_0$ holds. Put for $m = 1, 2, \dots$, $A_n = \{t : |x_n(t)| \geq u_0/2^{m-1}\}$, $B_n = (-\infty, \infty) - A_n$. The following inequalities are satisfied: $\mathfrak{I}_\varphi(h_{B_n} 2^m x_n) \leq c^m \mathfrak{I}_\varphi(h_{B_n} x_n) \leq c^m \mathfrak{I}_\varphi(x_n)$, $\mathfrak{I}_\varphi(h_{A_n} 2^m x_n) \leq \varphi(2^m) |A_n|$, $|A_n| \varphi(u_0/2^{m-1}) \leq \mathfrak{I}_\varphi(x_n)$. Thus (b) implies

$$(5) \quad \mathfrak{I}_\varphi(2^m x_n) = \mathfrak{I}_\varphi(h_{B_n} 2^m x_n) + \mathfrak{I}_\varphi(h_{A_n} 2^m x_n) \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

*) Compare e. g. [2].

Write $\varrho_x = \max [1, \mathfrak{I}_\varphi(x)]$. It follows from the Young inequality [1] that for $\mathfrak{I}_\varphi(y) \leq 1$, $\mathfrak{I}(xy\varrho_x^{-1}) \leq \mathfrak{I}_\varphi(x\varrho_x^{-1}) + \mathfrak{I}_\varphi(y) \leq 2$. Hence $\mathfrak{I}(xy) \leq 2\varrho_x$, thus $\|x\|^* \leq \|x\|^\circ \leq 2\varrho_x$. If we substitute in the last inequality $2^m x_n$ instead of x then we obtain from (5), $\|x_n\|^* \leq 2^{-m}$ for sufficiently large n . Hence $\|x_n\|^* \rightarrow 0$; thus (b) implies (a). Conversely, (a) implies (b) even without assumption (Δ_2) for small u . This follows from inequality $\mathfrak{I}_\varphi(x) = \mathfrak{I}_\varphi[\|x\|^* x (\|x\|^*)^{-1}] \leq \|x\|^*$ valid for $\|x\|^* \leq 1$.

Necessity. If condition (Δ_2) is not satisfied then there exists a sequence $u_n \downarrow 0$ ($u_1 < 1$) such that $\varphi(2u_n) \geq 2^n \varphi(u_n)$. We define a function $x_n(t)$ by $x_n(t) = u_n$ for $t \in \delta_n$, δ_n being an interval, $x_n(t) = 0$ for $t \notin \delta_n$, where $|\delta_n|^{-1} = \varphi(u_n)/2^{n/2}$. Evidently, $\mathfrak{I}_\varphi(x_n) \rightarrow 0$, $\|x_n\| \leq 1$, but $\mathfrak{I}_\varphi(2x_n) \rightarrow \infty$. Hence $\|2x_n\|^* \geq 1$ for almost all n .

1.5. THEOREM 3. *The space $M_s^{\mathfrak{I}_\varphi}$ is separable if, and only if, $\varphi(u)$ satisfies the condition (Δ_2) for small u .*

Sufficiency follows trivial from 1.4. Now, let us suppose that (Δ_2) for small u is not satisfied. Let us consider the sequence u_n and the functions $x_n(t)$ defined on the end of the proof of 1.4 with disjoint intervals δ_n . Then we have $\|z' - z''\|^* \geq 1/2$ for two different arbitrary functions z' , z'' of the form $z = \sum \eta_n x_n(t)$ (where $\eta_n = 0, 1$), which evidently belong to $M^{\mathfrak{I}_\varphi}$.

2. We say that a distributive functional ξ over $M^{\mathfrak{I}_\varphi}$ is linear over $M_s^{\mathfrak{I}_\varphi}$ if $\|x_n\| \leq 1$, and $\|x_n\|^* \rightarrow 0$ for $n \rightarrow \infty$ imply $\xi(x_n) \rightarrow 0$. We denote by $\|\xi\|^*$ (norm of the functional ξ over $M^{\mathfrak{I}_\varphi}$) the value $\sup |\xi(x)|$, where the supremum is taken over the set of elements $x \in M^{\mathfrak{I}_\varphi}$ such that $\|x\| \leq 1$ and $\|x\|^* \leq 1$.

2.1. THEOREM 4. *Functionals of the form*

$$(\alpha) \quad \xi_1(x) = \mathfrak{I}(xy_1), \quad y_1 \in L^1; \quad (\alpha') \quad \xi_2(x) = \mathfrak{I}(xy_2), \quad y_2 \in L^{\mathfrak{I}_\varphi}$$

are linear over $M_s^{\mathfrak{I}_\varphi}$.

(β) *If $\varphi(u)$ satisfies the condition (Δ_2) for small u then the general form of the linear functional over $M_s^{\mathfrak{I}_\varphi}$ is the following:*

$$(*) \quad \xi(x) = \xi_1(x) + \xi_2(x),$$

where ξ_1 and ξ_2 are functionals defined above by (α) and (α'), respectively. The functionals ξ_1 and ξ_2 appearing in the representation (*) of ξ may be always chosen so that

$$\|\xi\|^* = \|\xi_1\|^* + \|\xi_2\|^*, \quad \|\xi_1\|^* = \|y_1\|^*, \quad \|y_2\|^*/2 \leq \|\xi_2\|^* \leq 2\|y_2\|^*.$$

Let us remark that if the condition (Δ_2) for small u is not satisfied then there exist linear functionals over $M_s^{\mathfrak{I}_\varphi}$ not representable in the form of an integral $\mathfrak{I}(xy)$.

) $\|y_1\|^$ denotes the usual norm of the function in L^1 .

2.2. THEOREM 5. *Let $\varphi(u)$ satisfy the condition (Δ_2) for small u . Then a sequence ξ_n of linear functionals over M_s^{sp} is convergent for every $x \in M^{sp}$ if, and only if, the following conditions are satisfied:*

(a) *there exist functions $y_{1n} \in L^1$ and $y_{2n} \in L^{sp}$ such that for a certain constant $K > 0$, $\|y_{1n}\|^* \leq K$ and $\|y_{2n}\|^* \leq K$ for $n = 1, 2, \dots$,*

(b) *the functions y_{1n} and y_{2n} introduced in (a) generate the representation*

$$\xi_n(x) = \xi_1^n(x) + \xi_2^n(x), \quad n = 1, 2, \dots,$$

where

$$\xi_1^n(x) = \mathfrak{I}(xy_{1n}), \quad \xi_2^n(x) = \mathfrak{I}(xy_{2n});$$

(c) *the integrals $\int |y_n(t)| dt$ (where $y_n = y_{1n} + y_{2n}$) are equicontinuous over the ring of all sets of finite measure;*

(d) *if x_τ is the characteristic function of the interval $\langle 0, \tau \rangle$ then there exists $\lim_{n \rightarrow \infty} \mathfrak{I}(x_\tau y_n)$ for every τ .*

Theorems 4 and 5 are generalisations of theorems given in [4] and their proofs are analogical to them, too.

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Remarks on Quasi-Boolean Algebras

by

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Presented by A. MOSTOWSKI on April 1, 1957

This paper is a continuation of an earlier paper [1], the knowledge of which is here assumed. The terminology and notations in this paper are the same as in [1].

1. Remarks on congruences in distributive lattices

Let $\mathfrak{A} = \langle A, +, \cdot \rangle$ be always a distributive lattice *) with the unit element 1 and the zero element 0.

Let $i \subset A$ be a fixed ideal of \mathfrak{A} . We shall consider the following relations in the set A :

$$(a \underset{\min}{\simeq} b \pmod{i}) = \sum_{x \in i} (a + x = b + x),$$

$$(a \underset{\max}{\simeq} b \pmod{i}) = \prod_{x \in A} ((ax \in i) \Rightarrow (bx \in i)).$$

It is easy to verify that the following statement (cf. [3] Th. 1) holds:

1.1. *The relations $\underset{\min}{\simeq}$ and $\underset{\max}{\simeq}$ are the least and the greatest elements respectively in the lattice $\mathcal{C}(\mathfrak{A}, i)$ of all congruences \simeq in \mathfrak{A} , such that $E(x \simeq 0) = i$ **).*

For every $a \in A$ let $i_a = E_{x \in A}(ax = 0)$ and $i_a^* = E_{x \in A}(a + x = 1)$. We shall say that a distributive lattice \mathfrak{A} has the *disjunction property* ([4] p. 115) if

$$(i_a = i_b) \rightarrow (a = b) \quad \text{for any } a, b \in A.$$

Let h be a homomorphism of the lattice \mathfrak{A} in a distributive lattice \mathfrak{B} . A congruence \simeq on A is said to be *determined by h* if $a \simeq b$ if, and only if, $h(a) = h(b)$ for any $a, b \in A$.

*) For the basic notions of the theory of distributive lattice see e. g. [2].

**) By $E_{x \in X}(\dots)$, where the dots in the parentheses stand for a formula involving x , we mean the set of all elements x in X which satisfy this formula.

An ideal i of \mathfrak{A} is said to be determined by h , provided $i = \bigcup_{x \in A} (h(x)=0)$.

It is easy to see that

1.2. If the congruence determined by a homomorphism h is equal to the congruence $\simeq_{\max} \in \mathcal{C}(\mathfrak{A}, i)$, where i is the ideal determined by h , then $h(A)$ has the disjunction property.

In fact, if $h(a) \neq h(b)$, then there exists $x \in A$ such that either $ax \in i$ and $bx \notin i$ or $ax \notin i$ and $bx \in i$. Suppose that $ax \in i$ and $bx \in i$. Then $i_{h(a)} \neq i_{h(b)}$. Moreover,

1.3. If h is a homomorphism of lattice \mathfrak{A} , such that the lattice $h(A)$ has the disjunction property, then the congruence determined by h is equal to the congruence $\simeq_{\max} \in \mathcal{C}(\mathfrak{A}, i)$, where i is the ideal determined by h .

This remark follows from the fact that the congruence determined by h is greater than $\simeq_{\max} \in \mathcal{C}(\mathfrak{A}, i)$ (where i is the ideal determined by h).

2. Theorems on quasi-Boolean algebras

Let $\langle \{0, 1, a, b\}, +, \cdot \rangle$ be a distributive lattice such that $a + b = 1$. We define in $\{0, 1, a, b\}$ a unary operation as follows

$$\begin{aligned}\sim 0 &= 1, & \sim a &= a; \\ \sim 1 &= 0, & \sim b &= b.\end{aligned}$$

It is easy to see that $\mathfrak{A}_0 = \langle \{0, 1, a, b\}, +, \cdot, \sim \rangle$ is a quasi-Boolean algebra [1]. We shall denote by \mathfrak{B}_0 and \mathfrak{C}_0 respectively, the following subalgebras: $\langle \{0, 1\}, +, \cdot, \sim \rangle$, $\langle \{0, 1, a\}, +, \cdot, \sim \rangle$ of \mathfrak{A}_0 .

In the sequel, $\mathfrak{A} = \langle A, +, \cdot, \sim \rangle$ will always denote a quasi-Boolean algebra.

The following theorem can be easily obtained from the representation theorem of quasi-Boolean algebras proved in [1].

2.1. Every quasi-Boolean algebra is isomorphic with a subalgebra of product $\prod_{i \in Z} \mathfrak{A}_i$, where Z is a set of indices and $\mathfrak{A}_i = \mathfrak{A}_0$ for every $i \in Z$.

It is sufficient to prove that every quasi-field of sets (in the sense used in [1]) is isomorphic with a subalgebra of product $\prod_{i \in Z} \mathfrak{A}_i$.

Let $\langle A, \cap, \cup, \sim \rangle$ be a quasi-field of sets with the unit element $1 = U$ and let f be a mapping of U onto U such that $ff(u) = u$ for any $u \in U$ and $\sim X = U - f(X)$ for every $X \in A$.

If $x \in \prod_{i \in Z} \mathfrak{A}_i$ and $k \in Z$, then the element $x(k)$ will also be denoted by x_k .

We put $Z = \bigcup_{\langle x, y \rangle \in U \times U} (y = f(x))$ and then define a mapping h of A into $\prod_{i \in Z} \mathfrak{A}_i$ as follows:

$$(h(z))_{\langle x, f(x) \rangle} = \begin{cases} 0, & \text{when } \{x, f(x)\} \cap z = \emptyset^*, \\ a, & \text{when } \{x, f(x)\} \cap z = \{x\}, \\ b, & \text{when } \{x, f(x)\} \cap z = \{f(x)\}, \\ 1, & \text{when } \{x, f(x)\} \cap z = \{x, f(x)\}, \end{cases} \quad \text{for every } z \in A.$$

It is easy to verify that h is an isomorphism of $\langle A, \cup, \cap, \sim \rangle$ into $\mathcal{P}\mathfrak{A}_i$. As an immediate consequence of 2.1. we obtain the following

2.2. \mathfrak{A}_0 is a functionally free quasi-Boolean algebra.

Let $w_1(x_1, \dots, x_n)$ and $w_2(x_1, \dots, x_n)$ be any polynomials of quasi-Boolean algebra. We shall show that if there exists a quasi-Boolean algebra \mathfrak{A} in which $w_1(x_1, \dots, x_n)$ and $w_2(x_1, \dots, x_n)$ are not identically equal, then $w_1(x_1, \dots, x_n)$ and $w_2(x_1, \dots, x_n)$ are not identically equal in \mathfrak{A}_0 . We can suppose that \mathfrak{A} is a subalgebra of $\mathcal{P}\mathfrak{A}_i$ where $\mathfrak{A}_i = \mathfrak{A}_0$ for every $i \in Z$.

Then w_1 and w_2 are not identically equal in $\mathcal{P}\mathfrak{A}_i$. If $w_1(a_1, \dots, a_n) \neq w_2(a_1, \dots, a_n)$ for $a_1, \dots, a_n \in \mathcal{P}\mathfrak{A}_i$, then there exists $i \in Z$ such that $(w_1(a_1, \dots, a_n))_i \neq (w_2(a_1, \dots, a_n))_i$. Hence, $w_1((a_1)_i, \dots, (a_n)_i) \neq w_2((a_1)_i, \dots, (a_n)_i)$.

The algebra \mathfrak{C}_0 is not a functionally free quasi-Boolean algebra.

In fact, it is easy to verify that the polynomials

$$\begin{aligned} w_1(x_1, x_2, x_3, x_4) &= ((x_1 + \sim x_1) \cdot (x_2 + \sim x_2)) \cdot \sim((x_3 + \sim x_3) \cdot (x_4 + \sim x_4)), \\ w_2(x_3, x_4) &= \sim((x_3 + \sim x_3)(x_4 + \sim x_4)), \end{aligned}$$

are identically equal in \mathfrak{C}_0 but are not equal in \mathfrak{A}_0 .

An ideal p (a filter q) of \mathfrak{A} is said to be \sim ideal (\sim filter) provided that if $a \in p$ ($a \in q$) and $\sim ac \in p$ ($\sim a + c \in q$), then $c \in p$ ($c \in q$).

It is easy to see that

2.3. If h is a homomorphism of \mathfrak{A} into a quasi-Boolean algebra, then the set $\bigcup_{x \in A} (h(x) = 0)$ is a \sim ideal.

Let $i \subset A$ be a \sim ideal of \mathfrak{A} . We shall consider the following relations on the set A :

$$(a \approx_{\min} b \pmod{i}) = \left(\sum_{x \in i} (a + x = b + x) \wedge \sum_{x \in \bar{i}} (ax = bx) \right),$$

$$\text{where } \bar{i} = E \left(\sum_{x \in A} (x = \sim y) \right),$$

$$(a \approx_{\max} b \pmod{i}) = \left(\prod_{x \in A} ((ax \in i) \Rightarrow (bx \in i)) \wedge \prod_{x \in A} ((a + x \in \bar{i}) \Rightarrow (b + x \in \bar{i})) \right).$$

*) \emptyset denotes here the empty set.

Obviously, the relations \approx_{\min} and \approx_{\max} are the least and the greatest element respectively in the lattice $\mathcal{C}(\mathfrak{A}, i)$ of all congruences \approx in \mathfrak{A} , such that $\bigcup_{x \in A} (x \approx 0) = i$.

A quasi-Boolean algebra \mathfrak{A} is called *simple*, when the only \sim -ideal of \mathfrak{A} is the trivial one, i. e., formed from the element $0 \in A$.

As an immediate consequence of this definition we obtain

2.4. *A quasi-Boolean algebra \mathfrak{A} is simple if, and only if, it contains no non-trivial \sim -filter, i. e., the only \sim -filter of \mathfrak{A} is formed from the element $1 \in A$.*

2.5. *Every simple quasi-Boolean algebra is isomorphic with a subalgebra of \mathfrak{A}_0 .*

Let $\mathfrak{A} = \langle A, +, \cdot, \sim \rangle$ be a simple quasi-Boolean algebra. By 2.1. it is isomorphic with a subalgebra $\mathfrak{B} = \langle B, +, \cdot, \sim \rangle$ of $\mathcal{P}\mathfrak{A}_i = \mathfrak{A}$. For every $k \in Z$ the set $\bigcup_{x \in B} (x_k = 0) \left(\bigcup_{x \in B} (x_k = 1) \right)$ is a \sim -ideal (a \sim -filter). Hence, for every $x \in B$ if there exists $k \in Z$ such that $x_k = 0$ ($x_k = 1$), then $x = 0 \in B$ ($x = 1 \in B$)^{*}. If $B = \{0, 1\}$, then $\langle B, +, \cdot, \sim \rangle$ is isomorphic with \mathfrak{B}_0 . Otherwise there exists an element $a \in B$ such that $a_i \neq 0$ and $a_i \neq 1$ for every $i \in Z$. We may suppose that $a_i = a$ for every $i \in Z$. If $B = \langle 0, 1, a \rangle$, then $\langle B, +, \cdot, \sim \rangle$ is isomorphic with \mathfrak{C}_2 ; otherwise there exists an element $b \in B$ such that $b_k = b$ for some $k \in Z$. Hence we infer that $b_i = b$ for every $i \in Z$. If $B = \{0, 1, a, b\}$, then $\langle B, +, \cdot, \sim \rangle$ is isomorphic with \mathfrak{A}_0 . It is easy to see that B cannot contain more than these four elements.

We shall say that a quasi-Boolean algebra \mathfrak{A} has the *binary disjunction property* if

$$(\langle i_a, i_a^* \rangle = \langle i_b, i_b^* \rangle) \rightarrow (a = b) \quad \text{for any } a, b \in A.$$

It is easy to see that

2.6. *If the congruence determined by a homomorphism h is equal to the congruence $\approx_{\max} \in \mathcal{C}(\mathfrak{A}, i)$, where i is the \sim -ideal determined by h , then $h(A)$ has the binary disjunction property.*

2.7. *If h is a homomorphism of quasi-Boolean algebra \mathfrak{A} such that the quasi-Boolean algebra $h(A)$ has the binary disjunction property, then the congruence determined by h is equal to the congruence $\approx_{\max} \in \mathcal{C}(\mathfrak{A}, i)$, where i is the \sim -ideal determined by h .*

2.8. *Every simple quasi-Boolean algebra has the binary disjunction property.*

This follows from 2.5.

Moreover,

^{*}) 0 and 1 denote the unit element and the zero element of \mathfrak{A} respectively.

2.9. If h is a homomorphism of a quasi-Boolean algebra $\mathfrak{A} = \langle A, +, \cdot, \sim \rangle$ and the \sim -ideal $i = \bigcup_{x \in A} (h(x) = 0)$ is a maximal \sim -ideal then $h(A)$ is a simple quasi-Boolean algebra and consequently $h(A)$ has the binary disjunction property.

2.10. Let h_1 and h_2 be any homomorphism of \mathfrak{A} . If the set $i = \bigcup_{x \in A} (h_1(x) = 0) = \bigcup_{x \in A} (h_2(x) = 0)$ is a maximal \sim -ideal then $h_1(A)$ is isomorphic with $h_2(A)$.

This follows from 2.7. and 2.9.

Let us denote for every prime ideal i_0 by $g(i_0)$ the set $A - \tilde{i}_0$.

2.11. If i is a maximal \sim -ideal then there exists exactly one prime ideal i_0 such that $i = i_0 \cap g(i_0)$.

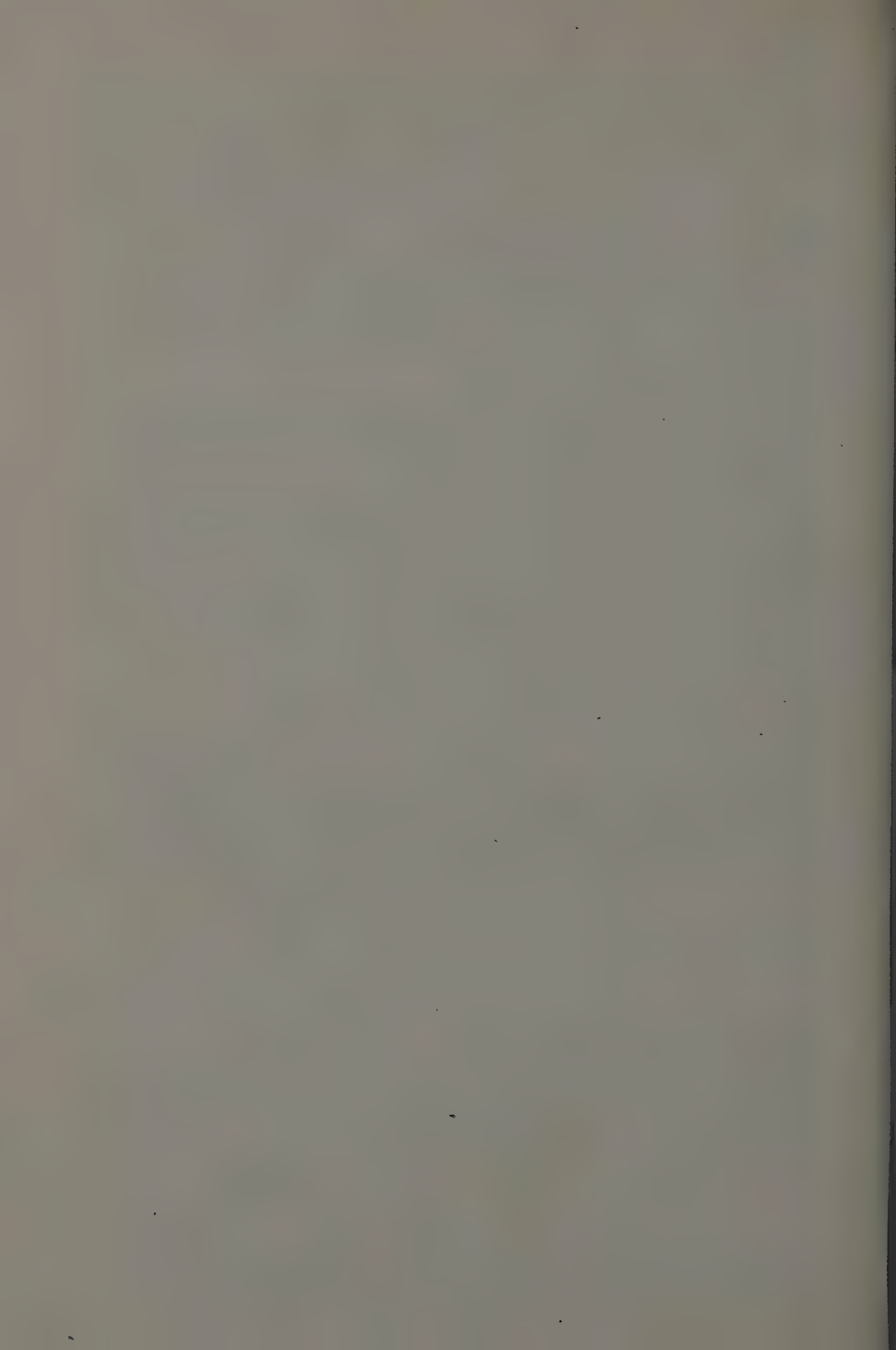
In fact, if i is a maximal \sim -ideal, then there exists a homomorphism h of \mathfrak{A} into \mathfrak{A}_0 such that i is determined by h . It is easy to verify that the set $h^{-1}(\{0, a\})$ is a prime ideal, $g(h^{-1}(\{0, a\})) = h^{-1}(\{0, b\})$ and $i = h^{-1}(\{0\}) = h^{-1}(\{0, a\}) \cap h^{-1}(\{0, b\})$.

The author wishes to express his gratitude to Mrs. H. Rasiowa for her advice in the preparation of this paper.

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Sur le problème d'oscillation des intégrales de l'équation

$$y'' + g(t)y = 0$$

par

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Présenté par T. WAŻEWSKI le 1 Avril, 1957

Dans la présente note nous indiquons plusieurs conditions qui garantissent que toutes les solutions de l'équation (4.1) sont oscillantes. A côté d'une condition nécessaire et suffisante (Théorème 1 de 10) nous obtenons quelques conditions suffisantes. Dans certaines de celles-ci, particulièrement simples, intervient la notion d'une limite approximative (Théorème 3 et 6). A cette notion sont consacrés les paragraphes 1 et 2. Dans le par. 3 nous indiquons quelques conditions suffisantes à ce qu'une telle limite approximative coïncide avec la limite classique. Ceci conduit au Théorème 4 dans lequel intervient la limite (resp. limite inférieure) au sens classique. Du Théorème 4 on peut déduire le théorème remarquable de Piétopavlovskaja ([1], Théorème 1).

1. Nous désignons par $\{t: W(t)\}$ l'ensemble des t vérifiant la condition $W(t)$, et par $m(E)$ la mesure de Lebesgue de l'ensemble E .

Soit $G(t)$ une fonction continue dans l'intervalle

$$\Delta = [0, +\infty).$$

Définition 1. Soit $-\infty \leq l \leq +\infty$, $-\infty \leq L \leq +\infty$. Nous dirons que

$$\lim_{t \rightarrow +\infty} \text{appr inf } G(t) = l,$$

lorsque

$$\begin{aligned} m\{t: G(t) \leq l_1\} &< +\infty && \text{pour tout } l_1 < l, \\ m\{t: G(t) \leq l_2\} &= +\infty && \text{pour tout } l_2 > l. \end{aligned}$$

Nous dirons que

$$\lim_{t \rightarrow +\infty} \text{appr sup } G(t) = L,$$

lorsque

$$\begin{aligned} m\{t: G(t) \geq L_1\} &= +\infty && \text{pour tout } L_1 < L, \\ m\{t: G(t) \geq L_2\} &< +\infty && \text{pour tout } L_2 > L. \end{aligned}$$

Pour $G(t)$ continue dans Δ il existe une seule limite inférieure l et une seule supérieure L .

Nous dirons que

$$(1.1) \quad \lim_{t \rightarrow +\infty} \text{appr } G(t) = \lambda,$$

lorsque $l = L = \lambda$.

La relation (1.1) a lieu alors (et alors seulement) qu'il existe un ensemble E tel que $E \subset \Delta$, $m(E) < +\infty$ et que $G(t)$ envisagé exclusivement sur $\Delta - E$ admet la limite classique $\lim_{t \rightarrow +\infty} G(t) = \lambda$.

2. Introduisons les limites classiques:

$$\liminf_{t \rightarrow +\infty} G(t) = l^*, \quad \limsup_{t \rightarrow +\infty} G(t) = L^*, \quad \lim_{t \rightarrow +\infty} G(t) = \lambda^*.$$

Proposition 1. On a évidemment

$$(2.1) \quad l^* \leq l \leq L \leq L^*.$$

3. Proposition 2. On a (cf. par. 1 et 2)

$$l^* = l, \quad L^* = L$$

lorsque la fonction $G(t)$ continue dans Δ remplit une des conditions A, B, C, D suivantes.

Condition A. À tout $\varepsilon > 0$ correspond un $\delta > 0$ tel que pour tout $t^* \geq \delta$, chacune des inégalités

$$G(t) \leq G(t^*) + \varepsilon, \quad G(t) \geq G(t^*) - \varepsilon$$

a lieu pour tous les t appartenant à un des intervalles fermés adjacents

$$[t^* - \delta, t^*], \quad [t^*, t^* + \delta].$$

Condition B. $G(t)$ est uniformément continue dans Δ (ce qui a lieu par exemple au cas où $G(t)$ satisfait à la condition de Hölder ou à celle de Lipschitz).

Condition C. $G'(t) \leq k < +\infty$ pour tout $t \in \Delta$ (k fixe).

Condition D. $G'(t) \geq m > -\infty$ pour tout $t \in \Delta$ (m fixe).

Démonstration. Chacune des conditions B, C, D implique la condition A. Admettons A et supposons que $l^* < l$ (cf. (2.1)). Pour un $\varepsilon > 0$ on a $l^* < l^* + 2\varepsilon < l$. Pour une suite $t_n \rightarrow +\infty$ on a $G(t_n) < l^* + \varepsilon$. Dans un des intervalles $[t_n - \delta, t_n]$, $[t_n, t_n + \delta]$ que nous désignons par θ_n on a $G(t) \leq G(t_n) + \varepsilon < l^* + 2\varepsilon < l$. En posant $\theta = \sum \theta_n$ on a $m(\theta) = +\infty$ et $G(t) < l^* + 2\varepsilon < l$ pour $t \in \theta$, ce qui contredit à la définition de l . On a donc $l^* = l$. On prouve pareillement que $L^* = L$.

4. Relativement à l'équation

$$(4.1) \quad y'' = -g(t)y$$

nous admettons

L'HYPOTHÈSE H. $g(t)$ est continue dans l'intervalle $\Delta = [0, +\infty)$.

Notation. Nous introduisons la fonction

$$(4.2) \quad G(t) = \int_0^t g(s) ds.$$

5. La condition nécessaire et suffisante pour qu'une intégrale $y = y(t)$ de (4.1) soit non banale, c'est-à-dire non identiquement nulle, consiste évidemment en ce que l'on ait

$$(5.1) \quad [y(t)]^2 + [y'(t)]^2 > 0 \quad \text{pour} \quad t \in \Delta.$$

6. Nous remplaçons (4.1) par le système équivalent

$$(6.1) \quad y' = u, \quad u' = -g(t)y$$

auquel nous appliquons consécutivement les transformations (cf. (4.2))

$$(6.2) \quad y = y, \quad u = -G(t)y + v,$$

$$(6.3) \quad y = r \cos \varphi, \quad v = r \sin \varphi.$$

Nous obtenons ainsi le système

$$(6.4) \quad r\varphi' = -r[\sin \varphi - G(t) \cos \varphi]^2,$$

$$(6.5) \quad r' = -r[G(t) \cos^2 \varphi + \{[G(t)]^2 - 1\} \cos \varphi \sin \varphi - G(t) \sin^2 \varphi].$$

7. Pour les intégrales non banales de (4.1) on aura (cf. (5.1), (6.2)

(6.3)) $r(t) \neq 0$ pour $t \in \Delta$ et, par suite (cf. (6.4)) on aura

$$(7.1) \quad \varphi' = -[\sin \varphi - G(t) \cos \varphi]^2,$$

$$(7.2) \quad \varphi' \leq 0 \quad (\text{pour } t \in \Delta).$$

En posant

$$(7.3) \quad \delta(t, \varphi) = [\sin \varphi - G(t) \cos \varphi]^2,$$

$$(7.4) \quad \gamma(t) = \arctg G(t),$$

on aura

$$(7.5) \quad \delta(t, \varphi) = (1 + [G(t)]^2) \{\sin(\varphi - \gamma(t))\}^2 \geq 0,$$

et l'on pourra écrire (7.1) sous la forme

$$(7.1 \text{ bis}) \quad \varphi' = -\delta(t, \varphi).$$

8. Toute solution φ de (7.1) ou (7.1 bis) existe dans Δ , décroît et par suite tend vers une limite m (dépendant de φ).

$$(8.1) \quad m = \lim_{t \rightarrow +\infty} \varphi(t).$$

9. En vertu de la périodicité $\delta(t, \varphi) \equiv \delta(t, \varphi + n\pi)$ ($n = \pm 1, \pm 2, \pm 3, \dots$) il suffit d'examiner l'équation (7.1 bis) dans la bande

$$(9.1) \quad -\frac{\pi}{2} \leq \varphi < \frac{\pi}{2}, \quad 0 \leq t < +\infty.$$

Si $\varphi = \varphi(t)$ est une solution de (7.1 bis) alors $\varphi = \varphi(t) + n\pi$ ($n = \pm 1, \pm 2, \dots$) l'est aussi. Il s'ensuit que, ou bien $m = -\infty$ (cf. (8.1)) pour toutes les solutions $\varphi(t)$ de (7.1 bis) ou bien il existe une solution pour laquelle

$$(9.2) \quad -\frac{\pi}{2} \leq m < \frac{\pi}{2}.$$

10. Une solution $y = y(t)$ de (4.1) est dite oscillante lorsqu'elle n'est pas banale (cf. par. 5) et lorsque l'équation $y(t) = 0$ ($t \geq 0$) admet une infinité de solutions en t . En s'appuyant sur les paragraphes 8 et 9 on démontre facilement le

THÉORÈME 1. *Chacune des conditions suivantes P et Q constitue une condition nécessaire et suffisante pour que toutes les intégrales non banales de (4.1) soient oscillantes:*

Condition P. Pour chaque solution $\varphi(t)$ de (7.1 bis) ou pour une au moins (ce qui revient au même dans le cas considéré) on a

$$\lim_{t \rightarrow \infty} \varphi(t) = -\infty.$$

Condition Q. Si

$$-\frac{\pi}{2} \leq m < \frac{\pi}{2},$$

la relation

$$\lim_{t \rightarrow \infty} \varphi(t) = m$$

ne peut avoir lieu pour aucune solution $\varphi = \varphi(t)$ de (7.1 bis).

11. La famille F de fonctions $\eta(t)$ sera composée des fonctions $\eta(t)$ continues, décroissantes au sens large dans Δ et telles que

$$\lim_{t \rightarrow \infty} \eta(t) = 0.$$

On a $\eta(t) \geq 0$ dans Δ lorsque la fonction $\eta(t) \in F$.

12. Soit

$$(12.1) \quad -\frac{\pi}{2} \leq m < \frac{\pi}{2}.$$

Introduisons les conditions $P_0(m) - P_4(m)$ suivantes

Condition $P_0(m)$. La relation

$$\lim_{t \rightarrow \infty} \eta(t) = m$$

n'a lieu pour aucune solution de (7.1 bis).

Condition $P_1(m)$. On a

$$\int_{-\infty}^{+\infty} \delta(t, m + \eta(t)) dt = +\infty \quad \text{lorsque la fonction } \eta(t) \in F.$$

Condition $P_2(m)$ au cas $-\frac{\pi}{2} < m < \frac{\pi}{2}$. On a

$$\int_{-\infty}^{+\infty} [\zeta(t) + \operatorname{tg}(m) - G(t)]^2 dt = +\infty \quad \text{lorsque la fonction } \zeta(t) \in F.$$

Condition $P_2(m)$ au cas où $m = -\frac{\pi}{2}$. On a

$$\int_{-\infty}^{+\infty} [1 + G(t)\zeta(t)]^2 dt = +\infty \quad \text{lorsque la fonction } \zeta(t) \in F.$$

Condition $P_3(m)$. La relation

$$\lim_{t \rightarrow +\infty} \operatorname{appr} \gamma(t) = m$$

(cf. (7.4) et par. 1) n'a pas lieu.

Condition $P_4(m)$. La relation

$$\lim_{t \rightarrow +\infty} \operatorname{appr} G(t) = \operatorname{tg}(m)$$

(où $\operatorname{tg}(-\pi/2) = -\infty$) n'a pas lieu.

LEMME 1. Admettons (12.1) pour m . Ceci posé on a les équivalences et implications logiques suivantes:

$$P_4(m) \equiv P_3(m) \supset P_2(m) \equiv P_1(m) \supset P_0(m).$$

13. THÉORÈME 2. Si pour chaque m remplissant les inégalités (12.1) est remplie au moins une des cinq conditions $P_i(m)$ ($i=0,1,2,3,4$), alors toutes les intégrales non banales de (4.1) sont oscillantes.

Démonstration. En vertu du Lemme 1 ce théorème résulte de la deuxième partie du Théorème 1 (avec la condition Q).

Voici un cas spécial bien simple du Théorème 2.

THÉORÈME 3. Si ou bien (cf. par. 1 et (4.2))

$$\lim_{t \rightarrow +\infty} \operatorname{appr} G(t) = +\infty,$$

ou bien cette limite approximative n'existe pas, alors toutes les intégrales non banales de (4.1) sont oscillantes.

THÉORÈME 4. Si la fonction $G(t)$ remplit dans Δ une des conditions A, B, C, D (cf. par. 3, proposition 2) et si, ou bien

$$\lim_{t \rightarrow \infty} G(t) = +\infty,$$

ou bien cette limite (au sens classique) n'existe pas, alors toutes les intégrales non banales de (4.1) sont oscillantes.

Démonstration. Le présent théorème résulte du Théorème 3 en vertu de la proposition du par. 3.

14. Posons $p^+(u)=u$ lorsque $u \geq 0$ et $p^+(u)=0$ lorsque $u < 0$.

THÉOREME 5. Si (cf. (4.2) et par. 11)

$$\int_0^{\infty} [1 + G(t)\zeta(t)]^2 dt = +\infty \quad \text{lorsque la fonction } \zeta(t) \in F,$$

$$\int_0^{\infty} [p^+(M - G(t))]^2 dt = +\infty \quad \text{pour tout } M \text{ fini},$$

alors toutes les intégrales non banales de (4.1) sont oscillantes.

Démonstration. Des hypothèses ci-dessus il résulte que la condition $P_2(m)$ est remplie pour tout m vérifiant (12.1). Notre théorème résulte donc du Théorème 2.

15. THÉOREME 6. Si (cf. par. 1 et (4.2))

$$(15.1) \quad \lim_{t \rightarrow +\infty} \text{appr inf } G(t) = l_0 \quad (l_0 \text{ fini}),$$

$$(15.2) \quad \int_0^{\infty} [p^+(l_0 - G(t))]^2 dt = +\infty,$$

alors toutes les solutions non banales de (4.1) sont oscillantes.

THÉOREME 7. Si

$$\lim_{t \rightarrow \infty} \inf G(t) = l_1^*, \quad (l_1^* \text{ fini}),$$

$$(15.3) \quad \int_0^{\infty} [p^+(l_1^* - G(t))]^2 dt = +\infty,$$

alors toutes les solutions non banales de (4.1) sont oscillantes.

Démonstration. Posons $m_0 = \text{arc tg } l_0$. De (15.2) il résulte que la condition $P_2(m)$ (cf. par. 12) est remplie pour $m_0 \leq m < \frac{\pi}{2}$. Il s'ensuit de (15.1) que la condition $P_4(m)$ est remplie pour $-\frac{\pi}{2} \leq m < m_0$. Le Théorème 6 résulte donc du Théorème 2. Pour la démonstration du Théorème 7 il suffit de remarquer que la condition (15.3) entraîne (15.2), car $l_1^* \leq l_0^*$ (cf. proposition 1 du par. 3).

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

OUVRAGES CITÉS

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A Necessary and Sufficient Condition for m -Almost-Metrisability

by

S. MRÓWKA

Presented by K. KURATOWSKI on April 16, 1957

We recall that a topological space X is said to be m -almost-metrisable provided that there exists a family $P = \{\rho_\xi\}_{\xi \in \Sigma}$ of pseudometrics on X of the power m such that $\bar{A} = \bigcap_{x \in X} \bigcap_{\rho_\xi \in P} \{\rho_\xi(x, A) = 0\}$, ($\rho_\xi(x, A)$ is the greatest lower bound of the numbers $\rho_\xi(x, y)$, where $y \in A$) for every $A \subset X$, or, equivalently, such that the sets $\bigcap_{y \in X} \{\rho_\xi(x, y) < \varepsilon\}$, ($x \in X$, $\rho_\xi \in P$, $\varepsilon > 0$) form a basis of X . In [1] it has been shown that a normal space is m -almost metrisable if, and only if, it has a basis which is the union of at most m locally finite systems (a system \mathfrak{R} of subsets of a topological space is said to be *locally finite* if every point of the space has a neighbourhood which intersects at most a finite number of members of \mathfrak{R}). An analogous result was obtained by Chan-Chen-Hon who has shown that in a normal space a uniformity having a basis of the power m can be introduced if, and only if, the space has a basis which is the union of at most m locally finite systems*). The two results are equivalent; indeed, a uniformity has a basis of the power m , if and only if, it is generated by a family of pseudometrics of the power m .

The purpose of the present paper is to give a necessary and sufficient condition for m -almost metrisability of completely regular spaces.

Definition 1. A system \mathfrak{R} of subsets of a topological space is said to be functionally locally finite (or functionally discrete) if there exists a system $\{f_U\}_{U \in \mathfrak{R}}$ of continuous functions each on the space to the unit interval $[0, 1]$ such that $f_U(x) = 1$ for $x \in U$ and the system $\{\bigcap_{x \in X} \{f_U(x) > 0\}\}_{U \in \mathfrak{R}}$ is locally finite (or discrete respectively). (A system of subsets of a to-

*) The result of Chan-Chen-Hon will be published in Doklady Akad. Nauk SSSR

topological space is said to be discrete if every point of the space has a neighbourhood which intersects at most one member of the system).

We shall now prove the following

THEOREM. *Let X be a completely regular space. Then the following conditions are equivalent:*

- (a) X is m -almost-metrisable;
- (b) X has a basis which is the union of m functionally discrete systems;
- (c) X has a basis which is the union of m functionally locally finite systems.

Let us introduce the following

Definition 2. A subset A of a topological space is said to be functionally contained in a subset B of the space (in symbols: $A \subset_f B$), if there is a continuous function on the space to the unit interval $[0,1]$ which is one on A and zero on the complement of B .

LEMMA. *If each member of a functionally locally finite system \mathfrak{R} of subsets of a topological space is functionally contained in a subset B of the space then the union of all members of \mathfrak{R} is also functionally contained in B .*

Proof. Let $\{f_U\}_{U \in \mathfrak{R}}$ be a system of functions mentioned in the definition of functionally locally finite systems. For every $U \in \mathfrak{R}$ there is a continuous function g_U which is one on U and zero on the complement of B . Let us set $h_U(x) = \min \{f_U(x), g_U(x)\}$. Clearly, h_U is one on U and zero on the complement of B and for $h_U(x) \leq f_U(x)$, the system $\{E(h_U(x) > 0)\}_{U \in \mathfrak{R}}$ is locally finite and it follows that the function $h(x) = \max_{U \in \mathfrak{R}} \{h_U(x)\}$ is continuous. But $h(x) = 1$ for $x \in \sum_{U \in \mathfrak{R}} U$, and $h(x) = 0$ for x belonging to the complement of B .

Proof of the theorem. 1° (a) implies (b). Suppose $P = \{\varrho_\xi\}_{\xi \in \Sigma}$ is a family of pseudometrics of the power m . Let \mathfrak{B}_n^ξ be the system of all the sets of the form $E_{y \in X}(\varrho_\xi(x, y) < 1/n)$ ($x \in X$, $\varrho_\xi \in P$, $n = 1, 2, \dots$) and \supset be a relation which well orders \mathfrak{B}_n^ξ . For every $U \in \mathfrak{B}_n^\xi$ and a positive integer k let $U_k = E_x(\varrho_\xi(x, X - U) \geq \frac{1}{2^k})$ and $\tilde{U}_k = U_k - \sum_{U' < U} U'_{k+1}$. Clearly, $\varrho_\xi(U_k, X - U_{k+1}) > \frac{1}{2^{k+1}}$ and $\tilde{U}_k \subset U_k \subset U$. Let us observe that if $U' < U$ then $U_k \cdot \tilde{U}'_{k+1} = 0$, hence $\tilde{U}_k \subset X - U'_{k+1}$. But $\varrho_\xi(U'_k, X - U'_{k+1}) \geq \frac{1}{2^{k+1}}$, and it follows that $\varrho_\xi(U'_k, U_k) \geq \frac{1}{2^{k+1}}$. For every $U \in \mathfrak{B}_n^\xi$ let $\check{U}_k = E_{x \in X} \left[\varrho_\xi(x, \tilde{U}_k) < \min \left\{ \frac{1}{n}, \frac{1}{2^{k+3}} \right\} \right]$, and let $\mathfrak{B}_{n,k}^\xi$ be the system of all \check{U}_k ($U \in \mathfrak{B}_n^\xi$). We have $\check{U}_k \supset U_k$, $\delta_\xi(\check{U}_k) < \frac{3}{n}$ *, $\varrho_\xi(\check{U}_k, U'_k) \geq \frac{1}{2^{k+2}}$ for every

*) $\delta_\xi(A)$ denotes the ξ -th pseudodiameter of A , i. e., the number $\sup_{x, y \in A} \varrho_\xi(x, y)$.

$\check{U}_k, \check{U}'_k \in \mathfrak{B}_{n,k}^\xi$ $U_k \neq U'_k$. Let us set $f_{\check{U}_k}(x) = \max \{0, 1 - 2^{k+4} \cdot \varrho_\xi(x, \check{U}_k)\}$. Clearly, $f_{\check{U}_k}(x) = 1$ for $x \in \check{U}_k$, and $\varrho_\xi \left[\bigcup_x \{f_{\check{U}_k}(x) > 0\}, \bigcup_x \{f_{\check{U}'_k}(x) > 0\} \right] \geq \frac{1}{2^{k+3}}$ for every $\check{U}_k, \check{U}'_k \in \mathfrak{B}_{n,k}^\xi$, $\check{U}_k \neq \check{U}'_k$ and it follows that the system $\{ \bigcup_x \{f_{\check{U}_k}(x) > 0\} \mid \check{U}_k \in \mathfrak{B}_{n,k}^\xi \}$ is discrete, hence the system $\mathfrak{B}_{n,k}^\xi$ is functionally discrete. We shall show that the union of all $\mathfrak{B}_{n,k}^\xi$ ($x \in \Xi$; $n, k = 1, 2, \dots$) is a basis of X . It suffices to show that for every $x \in X$ and $\xi \in \Xi$ and every positive integer n there exists a positive integer k and $U \in \mathfrak{B}_n^\xi$ such that $x \in \check{U}_k$. Let U be the first member of \mathfrak{B}_n^ξ which contains x . Then $x \in U_k$ for some k and because U is the first member of \mathfrak{B}_n^ξ containing x , $x \in U_k - \sum_{U' < U} U'_{k+1} = \check{U}_k$ and finally $x \in U_k$.

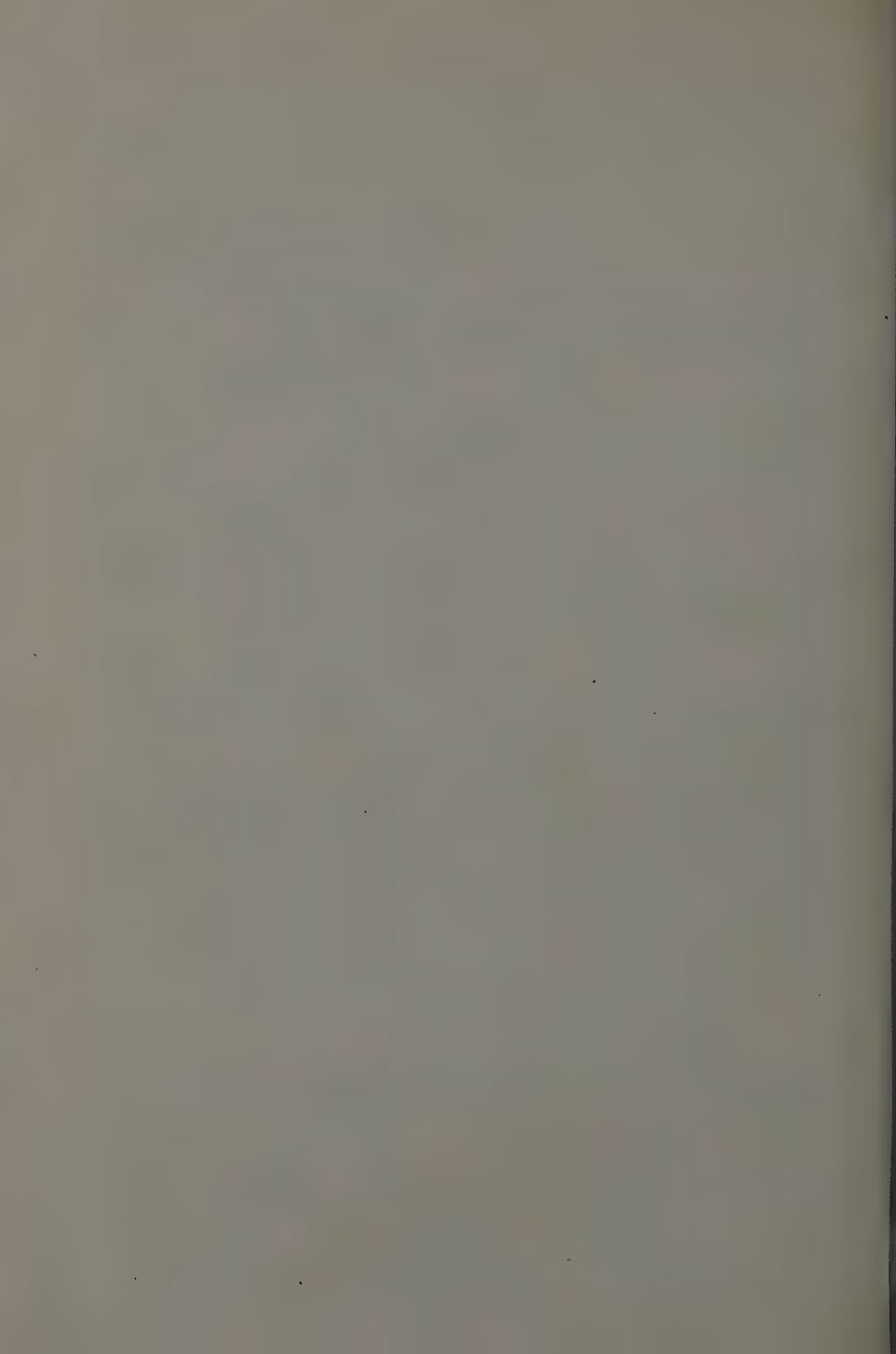
2° (b) implies (c). This implication is obvious because each functionally discrete system is functionally locally finite.

3° (c) implies (a). Let \mathfrak{B} be a basis of X such that $\mathfrak{B} = \sum_{\xi \in \Xi} \mathfrak{B}_\xi$ ($\Xi = m$), where \mathfrak{B}_ξ are functionally locally finite systems. For a given ordered pair $\langle \xi_1, \xi_2 \rangle \in \Xi \times \Xi$ and for every $U \in \mathfrak{B}_{\xi_2}$ let us denote as U' the union of all $V \in \mathfrak{B}_{\xi_1}$ such that $V \subset_f U$. By Lemma, $U' \subset_f U$, i. e., there is a continuous function f_u on X to the unit interval $[0, 1]$ which is one on U' and zero on $X - U$. Let us set $\varrho_{\xi_1, \xi_2}(x, y) = \sum_{U \in \mathfrak{B}_{\xi_2}} |f_U(x) - f_U(y)|$. The local finiteness of \mathfrak{B}_{ξ_2} implies the continuity of ϱ_{ξ_1, ξ_2} . Let P be the family of all ϱ_{ξ_1, ξ_2} , ($\xi_1, \xi_2 \in \Xi$). We shall show that P is the required family of pseudometrics. If $x_0 \in \bar{A}$, then by the continuity of ϱ_{ξ_1, ξ_2} , $\varrho_{\xi_1, \xi_3}(x_0, A) = 0$. If $x_0 \notin \bar{A}$, then there exists a neighbourhood U of x_0 belonging to some \mathfrak{B}_{ξ_2} such that $U \cdot \bar{A} = \emptyset$. There exists a continuous function ψ on X to the unit interval such that $\psi(x_0) = 1$ and $\psi(x) = 0$ for $x \in X - U$. There exists a neighbourhood V of x_0 belonging to some \mathfrak{B}_{ξ_1} such that $V \subset \bigcup_x \{ \varphi(x) > \frac{1}{2} \}$. Clearly, $V \subset_f U$, thus $V \subset U'$; hence $f_U(x_0) = 1$. But $f_U(x) = 0$ for $x \in A$ and it follows that $\varrho_{\xi_1, \xi_2}(x, A) \geq \inf_{y \in A} |f_U(x_0) - f_U(y)| = 1$.

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Note on the Two-Stage Mechanism of the (γ, d) Reactions

by

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Presented by L. INFELD on April 12, 1957

Investigations of (γ, n) , (γ, p) and (γ, d) reactions have, within the last few years, established the failure of the statistical theory to explain these reactions (see, e. g. [1]–[13]). The main bulk of the evidence is provided by the angular distributions of (γ, n) and (γ, p) reactions, and the ratios of (γ, d) to (γ, p) or (γ, n) reaction yields. The particularly drastic departure from the statistical model comprises the magnitudes of the (γ, d) yield, which is about a thousand times larger than those predicted from the model. The present paper sets out to present some remarks on this problem.

In the same way as suggested by Byerly and Stephens [2] and Toms and Stephens [9], the photodeuterons are assumed in the present treatment to be formed in the pick-up process within the nucleus — i. e., the two-stage mechanism of the reaction is supposed to be valid: the incident photon is absorbed by a single nucleon which on its subsequent way out picks up a second nucleon with the correct momentum to form a deuteron emerging from the nucleus. It is easy to advance a number of arguments why the probability of the pick-up process for a sufficiently energetic nucleon *within a nucleus* is large enough to explain the observed $\sigma(\gamma, d)/\sigma(\gamma, p)$ ratios.

A number of recent papers indicate that the Independent Particle Model (I. P. M.) of the nucleus satisfactorily explains photonuclear reactions (see, e. g., [14]–[17]). Also, a survey of the observed $\sigma(\gamma, d)/\sigma(\gamma, p)$ ratios (see, e. g., [11]) for various elements at intermediate energies reveals considerable fluctuations, seeming to indicate that the emission of photodeuterons is sensitive to the details of the nuclear shell structure rather than to the general type of the momentum distribution of nucleons in the nucleus *).

*) On the other hand, the correlations in the nuclear ground state wave function (in the spirit of, e. g., Chew and Goldberger [18], Brueckner, Eden, Francis [19],

Unfortunately, the existing data provide very scant information concerning angular distributions of photodeuterons which seem to be enlightening as regards the mechanism of the (γ, d) reaction. These facts suggest consideration of $\sigma(\gamma, d)/\sigma(\gamma, p)$ in a simplified case such as would, however, present a picture of the reaction based on I. P. M.

We shall calculate $\sigma(\gamma, d)/\sigma(\gamma, p)$ for a simplified system of two bound nucleons: the neutron and the proton (we shall first disregard the spins of the nucleons) are bound *via* the potentials $V_n(\vec{r}_n)$ and $V_p(\vec{r}_p)$, the mutual interaction being $V_{np}(\vec{r}_n - \vec{r}_p)$, and the interaction with the electromagnetic fields — $H_{\text{e.m.}}$. The total hamiltonian of the system $H = H_n + H_p + V_{np} + H_{\text{e.m.}}$, where $H_{n(p)} = -\frac{\hbar^2}{2M} \Delta_{n(p)} + V_{n(p)}$, M being the nucleon mass. Let us assume (it is unimportant here) that $H_{\text{e.m.}}$ interacts with only one nucleon (proton, say). If there were no V_{np} , the motions of both the nucleons would be independent. We might then write the wave function of the system in the form:

$$(1) \quad \Psi_0(\vec{r}_n, \vec{r}_p, t) = f(\vec{r}_n) \exp\left(-\frac{i}{\hbar} \epsilon t\right) \left[u(\vec{r}_p) \exp\left(-\frac{i}{\hbar} \epsilon_p t\right) + w(\vec{r}_p) \times \right. \\ \left. \times \exp\left[-\frac{i}{\hbar} (\epsilon_p + \hbar\omega) t\right] + v(\vec{r}_p) \exp\left[-\frac{i}{\hbar} (\epsilon_p - \hbar\omega) t\right] \right],$$

$\epsilon_{p(n)}$ being the binding energies of the proton (neutron), $\hbar\omega$ — the photon energy, $f(\vec{r}_n)$ — the wave function of the bound neutron, and the bracket term representing the wave function of the photoproton obtained in analogy to Sommerfeld and Schur [21]. From the three functions u , w , v , only w has a wave zone and, in our case, contributes; $u(\vec{r}_p)$ represent the wave function of the bound proton. The function w satisfies the differential equation [21]:

$$(2) \quad \Delta + w(k_f^2 - U) = w \Delta u,$$

where

$$k_f^2 = \frac{2M}{\hbar^2} (\epsilon_p + \hbar\omega), \quad U = \frac{2M}{\hbar^2} V_p,$$

$A = \frac{2M}{\hbar^2} E_0 e' \sqrt{\frac{\pi}{3}} r Y_{10}(\theta, \varphi)$, provided the incident photon wave is polarised along the z -axis and only electric dipole transitions are taken into consideration *); E_0 denotes the amplitude of the electric field, e' is the effective charge of nucleon.

Levinger [20] and many others — i. e., the high-energy components in the momentum distribution of the nucleons in the nucleus — may be important at high energies.

*) These transitions are, according to recent papers, notably dominant for nuclei (see, e. g., [16]).

Let us consider now the wave equation of the system $i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$.

We are seeking the solution in the form $\Psi = \Psi_0 + \Psi_1$. After simple transformations, we obtain for Ψ_1 the equation

$$(3) \quad \left(H - i\hbar \frac{\partial}{\partial t} \right) \Psi_1 = -V_{np}fu \exp\left(-\frac{i}{\hbar} \epsilon_0 t\right) - V_{np}fw \exp\left(-\frac{i}{\hbar} Et\right) - V_{np}fv \exp\left(-\frac{i}{\hbar} \epsilon_1 t\right),$$

where $\epsilon_0 = \epsilon_n + \epsilon_p < 0$, $\epsilon_1 = \epsilon_p + \epsilon_n - \hbar\omega < 0$, $E = \epsilon_p + \epsilon_n + \hbar\omega > 0$.

We look for Ψ_1 in the form:

$$\Psi_1 = \zeta \exp\left(-\frac{i}{\hbar} \epsilon_0 t\right) + \xi \exp\left(-\frac{i}{\hbar} \epsilon_1 t\right) + \chi \exp\left(-\frac{i}{\hbar} Et\right).$$

We obtain

$$(H - \epsilon_0)\zeta = -V_{np}fu, \quad (H - \epsilon_1)\xi = -V_{np}fv,$$

and

$$(4) \quad (H - E)\chi = -V_{np}fw.$$

Only this last equation has a wave zone and can represent the emission of photodeuterons. After changing the variables viz. $\frac{\vec{r}_n + \vec{r}_p}{2} = \vec{R}$, $\vec{r}_n - \vec{r}_p = \vec{\varrho}$, and disregarding $H_{c.m.}$ as compared with other terms, Eq. (4) takes on the form:

$$(5) \quad (H_d - T_R - E)\chi(\vec{R}, \vec{\varrho}) = -(V_n + V_p)\chi(\vec{R}, \vec{\varrho}) - V_{np}fw,$$

where H_d is the hamiltonian of the $n-p$ system, and T_R — the kinetic energy operator of the centre of mass of the deuteron. Let us develop χ in terms of $n-p$ system eigenfunctions:

$$(6) \quad \chi(\vec{R}, \vec{\varrho}) = D(\vec{R})\eta(\vec{\varrho}) + \text{orthogonal terms},$$

$\eta(\vec{\varrho})$ being the deuteron ground state wave function. On substituting (6) into (5) we have for $D(\vec{R})$:

$$(7) \quad (\Delta + K^2)D(\vec{R}) = \frac{4M}{\hbar^2} \int d\vec{\varrho} \eta^*(\vec{\varrho}) \left[V_n \left(\vec{R} + \frac{\vec{\varrho}}{2} \right) + V_p \left(\vec{R} - \frac{\vec{\varrho}}{2} \right) \right] \chi(\vec{R}, \vec{\varrho}) + \frac{4M}{\hbar^2} \int d\vec{\varrho} \eta^*(\vec{\varrho}) V_{np}(\vec{\varrho}) f \left(\vec{R} + \frac{\vec{\varrho}}{2} \right) w \left(\vec{R} - \frac{\vec{\varrho}}{2} \right),$$

where $K^2 = \frac{4M}{\hbar^2} (E - \epsilon_d)$ (ϵ_d = the binding energy of the deuteron). $D(\vec{R})$ is the wave function of the outgoing deuterons. The crudest approximation for $D(\vec{R})$ is obtained by setting $\chi(\vec{R}, \vec{\varrho}) = 0$ on the right-hand side of

Eq. (7) and assuming the zero-range n - p interaction — i. e., $\eta(\vec{q})V_{np}(\vec{q}) = -\frac{4\pi\hbar^2}{M}\sqrt{\frac{a}{2\pi}}\delta(\vec{q})$. Then

$$(8) \quad (\Delta + K^2)D^{(0)}(\vec{R}) = -16\pi\sqrt{\frac{a}{2\pi}}f(\vec{R})w(\vec{R}).$$

Eq. (8) determines the reaction amplitude in the Born approximation. Since the pick-up process occurs inside the nucleus, where both V_n and V_p are most important, the approximation of Eq. (8) is but very poor. On expanding $V_n + V_p$ in powers of \vec{q} , it can be shown that the first essential correction to the effective interaction $(V_n + V_p)_{q=0} = V_{\text{eff}}(\vec{R})$, being proportional to q^2 , is small. One is led, therefore, to the “distorted wave” approximation for Eq. (7):

$$(9) \quad \left(\Delta + K^2 - \frac{4M}{\hbar^2}V_{\text{eff}}\right)D(\vec{R}) = -16\pi\sqrt{\frac{a}{2\pi}}f(\vec{R})w(\vec{R}).$$

Eq. (9) neglects, of course, the effects of the deuteron dissociation in the process within the nucleus.

It is easy to obtain the angular distributions of the photodeuterons from Eq. (9), if the angular dependence of $w(\vec{R})$ is known and specific bound state wave functions f and u are assumed. For the sake of illustration let us assume f and u to represent S -states. Let us expand $w(\vec{R})$ of Eq. (2) in spherical harmonics:

$$w(\vec{R}) = \sum_{l,m} w_l(\vec{R}) Y_{lm}(\theta, \varphi).$$

Substituting into Eq. (2), and comparing the coefficients of the spherical harmonics, we obtain a series of differential equations for $w_l(R)$. Only the equation for w_1 is inhomogeneous since the expansion of the right-hand side contains a single spherical harmonic only: Y_{10} . Only this equation has, therefore, the solution satisfying the proper boundary conditions required, is regular at $R=0$, and produces a spherical divergent wave in infinity. All the remaining w_l satisfying homogeneous equations are to be set equal to zero, since they cannot satisfy the boundary conditions. The angular dependence of $w(\vec{R})$ is therefore $Y_{10}(\theta, \varphi)$. The angular dependence of $D(\vec{R})$ can be determined similarly from Eq. (9) after expanding $D(\vec{R})$ in spherical harmonics. Since f is, in our simple example, a spherically symmetric S -state, $D(\vec{R})$, also has the angular dependence $Y_{10}(\theta, \varphi)$. Thus, if the “photo-pickup” process is of the type (S, S) — i. e., if both the photo-effect and the pickup come from S -states — then (γ, p) and (γ, d) processes have the same angular distributions. It follows that the ratio $\sigma(\gamma, d)/\sigma(\gamma, p)$ is isotropic. The matter becomes complicated when the (γ, d) process is of the type (S, D) , (P, S) , (D, S) , etc.

We can then expand the right-hand side of Eq. (2) in spherical harmonics, making use of the formula

$$(10) \quad Y_{lm} Y_{10} = \sum_L \left[\frac{3(2L+1)}{4\pi(2L+1)} \right]^{1/2} C_n(L, 0; 0, 0) C_n(L, m; m, 0) Y_{Lm},$$

C_{ln} being the usual Clebsch-Gordon coefficients. We can now solve the inhomogeneous equations for w_l and find $w(\vec{R})$. $D(\vec{R})$ can then be found by Eq. (10), but the ratio $\sigma(\gamma, d)/\sigma(\gamma, p)$ will, in general, be no longer isotropic.

Let us pass over now to the construction of the photonucleon wave function $w(\vec{R}) = \sum_{l,m} w_l(R) Y_{lm}$. From Eq. (2) it is seen that w_l satisfies the equation

$$(11) \quad \frac{1}{R^2} \frac{d}{dR} \left(R^2 \frac{dw_l}{dR} \right) + \left(k_f^2 - U - \frac{l(l+1)}{R^2} \right) w_l(R) = (Au)_l.$$

The solution can be written in the form

$$(12) \quad w_l(R) = \int_0^\infty G_l(R/R') (Au)_l R'^2 dR',$$

the Green function being

$$(13) \quad G_l(R/R') = k_f e^{-i\eta_l} \begin{cases} F_l(k_f R') I_l^{(+)}(k_f R) & R' < R \\ F_l(k_f R) I_l^{(+)}(k_f R') & R' > R, \end{cases}$$

where F_l and $I_l^{(o)}$ are, respectively, the property normalised regular and irregular (outgoing) solutions of the homogeneous equation (see, e. g., [22] for neutrons). If the Coulomb potential (for protons) is included in U , we have to replace the respective expressions occurring in Eqs. (29)–(34) [22] by their Coulomb analogues — viz.

$$\begin{aligned} j_l(x) & \text{ by } e^{i\sigma_l(n)} F_l^c(n, x)/x \\ n_l(x) & \text{ by } e^{i\sigma_l(n)} G_l^c(n, x)/x, \end{aligned}$$

where $\sigma_l(n) = \arg I_l(l+1+in)$, n being the Gamow parameter of the proton. The phase-shift η_l , being δ_l for neutrons, is $\delta_l^c + \sigma_l(n)$ for protons. In view of Eq. (13), Eq. (12) can be rewritten in the form:

$$(14) \quad w_l(R) = k_f e^{-i\eta_l} \left\{ I_l^{(+)}(k_f R) \int_0^R F_l(k_f R') (Au)_l R'^2 dR' + \right. \\ \left. + F_l(k_f R) \int_R^\infty I_l^{(+)}(k_f R') (Au)_l R'^2 dR' \right\}.$$

Eq. (14) represents the photonucleon wave over all space (including the interior of the nucleus).

The same procedure can be applied to the photodeuteron wave function $D_l(R)$. The asymptotic form of $D_l(R)$ following from Eq. (9) is

$$(15) \quad D_l(R) \sim -16\pi \sqrt{\frac{\alpha}{2\pi}} R^{-1} \exp[i(kR - n_d \ln 2kR + \eta_l^d + \sigma_l(nd))] \times \\ \times \int_0^\infty \mathfrak{F}_l(kR) [f(\vec{R}) w(\vec{R})]_l R^2 dR,$$

$\mathfrak{F}_l(KR)$ being the properly normalised regular solution of the homogeneous equation

$$(16) \quad \left[\frac{1}{R^2} \frac{d}{dR} R^2 \frac{d}{dR} + K^2 + \frac{4M}{\hbar^2} V_{\text{eff}} - \frac{l(l+1)}{R^2} \right] \mathfrak{F}_l(KR) = 0.$$

The theory presented above can be applied, for instance, to the $^{32}\text{S}(\gamma, d)^{30}\text{P}$ reaction. According to the nuclear shell model, the most loosely bound are the four 2S-shell nucleons (2 protons + 2 neutrons). It may be assumed, therefore, that near the threshold of the reaction (19.15 MeV, see [3]), deuterons are produced mainly due to the (S, S) process, and hence, after averaging over the directions of the photon polarisation, the photodeuterons will have the angular distribution of the form $\sin^2 \theta$ — θ being the c. m. system angle between the direction of the incident γ -rays and the direction of the outgoing deuterons. The ratio $\sigma(\gamma, d)/\sigma(\gamma, p)$ can be calculated, the (γ, p) process being assumed to come mainly from S-states *):

$$(17) \quad \frac{\sigma(\gamma, d)}{\sigma(\gamma, p)} = \left(16\pi \sqrt{\frac{\alpha}{2\pi}} \right)^2 \frac{1}{2} K k_f \left| \int_0^\infty \mathfrak{F}_1(KR) [f(\vec{R}) w(\vec{R})]_1 R^2 dR \right|^2 \times \\ \times \left| \int_0^\infty F_1(k_f R) (Au)_1 R^2 dR \right|^{-2}.$$

The computation in the crudest case of Born approximation was improved by taking into account the effects of the Coulomb and nuclear interactions (of V_{eff} and V_p) by means of the barrier penetrabilities calculated on the method given by Dedrick [23]. For $\hbar\omega = 25$ MeV, the ratio of the penetrability factors is 0.53. The final result in the (S, S) case is $\sigma(\gamma, d)/\sigma(\gamma, p) = 0.9$. If $\sigma(\gamma, p)$ were improved by the consideration of other proton shells, this ratio might be reduced. In any case it is seen, however, that the number obtained may be viewed only as one more

*) Eq. (17) does not represent the experimental $\sigma(\gamma, d)/\sigma(\gamma, p)$ since it takes into account only the photoprotons from the S-shell, while the incident photon energy being > 19.15 MeV is sufficient to produce photoprotons from lower-lying shells.

very crude estimation indicating large values for the ratio $\sigma(\gamma, d)/\sigma(\gamma, p)^*$. For the energy $\hbar\omega = 25$ MeV, the (S, D) and (D, S) processes are in fact possible. They modify the angular distribution of photodeuterons from the $\sin^2 \theta$ type to the $a + b \sin^2 \theta$ type. The constants a and b can be calculated on the basis of methods outlined in the preceding sections.

The estimations obtained indicate that the pick-up mechanism can, in fact, explain the large $\sigma(\gamma, d)/\sigma(\gamma, p)$ values observed. The best way of checking this conclusion seems to be by way of measuring the angular distribution of photodeuterons — as, for instance, for the $^{32}\text{S}(\gamma, d)^{30}\text{P}$ reaction. Near the threshold of this reaction, the pick-up mechanism should produce the angular distribution $\sin^2 \theta$ according to the nuclear shell model. The method of calculation outlined may be used, in principle, for nuclei of more complicated shell structure. It seems that in such nuclei the correlation effects between nucleons due to the anti-symmetrisation of the wave function of particular shells are important. This should introduce the fractional parentage overlaps [17]. The fluctuations of the ratios $\sigma(\gamma, d)/\sigma(\gamma, p)$ observed [11], might be explained with the help of coefficients of fractional parentage. I. P. M. is probably not suitable at high energies to describe (γ, d) reactions (the two-body correlations are then important, [18], [19]).

Our thanks are due to J. Dąbrowski, who called our attention to the importance of the problem of large $\sigma(\gamma, d)/\sigma(\gamma, p)$ values. We would also like to thank Professors O. A. Hanson and L. Katz for a helpful correspondence on the problem, and Professor W. C. Parkinson for kindly sending us a microfilm of Dedrick's unpublished thesis.

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*) $\sigma(\gamma, p)$ in the (S, S) case was also calculated, taking into account U with no Coulomb effect in the final state, and found to be highly sensitive to the choice of U . A reasonable $\sigma(\gamma, d)/(\sigma, p)$ ratio can be obtained by properly adjusting U .

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Observations on Etching of Germanium Crystals

by

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Presented by A. SOZTAN on December 13, 1957

For investigations on the physical properties of Ge crystals, it is very important to know about the perfection on their structure and the type of defects occurring in them. One method of investigating the perfection of crystals is by etching their surfaces. Defects in the crystal lattice cause differences in the speed of the etching, and, as a result, etched figures arise.

The speed of etching increases with an increase in the deformation energy. The boundaries of grains slightly twisted with respect to one another appear during the etching in the form of separate points. Between dislocations, the etching process proceeds with more difficulty in the case of weak stresses than in regions of great stresses [1]—[4].

The investigations presented in this paper concern the dependence of the structure of the surfaces of germanium crystals on the manner of polishing and on the etching time. The crystals were grown both by Czochralski's technique and by zonal melting. The material used for the production of the crystals was purified several times by zonal melting. The purity was such that the specific resistivity was about 50 ohm·cm.

After the crystal was filed, a layer about 4 mm. thick was removed by polishing and etching. It was on a surface thus prepared that the secondary figures were observed. The natural growth faces were also investigated. After polishing with carborundum or alundum of different sized grains, a layer about 0.5 to 1 mm. thick was removed each time by rapid etching in concentrated CP-4. The surfaces prepared in this way were observed, and they were subjected to further slow etching in diluted CP-4 for 20 to 60 hours.

Of all the reagents tested, CP-4 proved to be the best etching agent. The orientation of the crystals were determined by the Laue method.

The surfaces of various germanium crystals were polished by carborundum of grain size $0.075 \leq d \leq 0.095$ mm. and etched for as much

as two hours in concentrated CP-4. Regardless of crystallographic orientation these surfaces showed a block structure (Figs. 1, 2, 3 — block size from 0.1 to 0.3 mm.; in Figs. 1 and 2 the surface investigated forms an angle of 7° with the (100) plane, while in Fig. 3 the angle formed was of 5° with the (111) plane).

After polishing with finer carborundum and alundum, it was noted that on some surfaces, already after 15 minutes of etching, the block structure disappeared and the surfaces became smooth. On other surfaces there was observed a block structure regardless of the etching time; the dimensions of the blocks were correspondingly smaller than when polished with coarse carborundum.

The size of the blocks decreased with the etching time, and after two hours was, on the average, from three to ten microns. In the case of surfaces with an orientation approximate to (100), the process of the disappearance of the block structure was accompanied by the appearance of "conical" etched figures of a mean diameter of $10\ \mu$ (Figs. 4, 5 — the surface under investigation formed an angle of 7° with the (100) plane).

With an increase in time, regardless of the manner of polishing (carborundum of various sized grains; however, $d < 0.075$ mm.), a further disappearance of the block structure and a sharpening of the "conical" figures were noted. In Fig. 6 are visible traces of the block structure and the sharp conical figures (the surface under investigation formed an angle of 7° with the (100) plane). Frequently one finds a certain order as regards the direction of the conical figures (Figs. 7 to 9 concern a surface forming an angle of 8° with the (100) plane). On some crystals a considerable inclination of the axes of the cones may be noted (Fig. 10, 11 — surface under investigation forms an angle of $7^\circ 30'$ with (100); Fig. 12, 13 — angle of 8° with (100)). The observed figures appear to be etched to a depth of $1\ \mu$.

In Fig. 14 may be seen the displacement of the lines of the conical figures (boundary of the dislocation) under the influence of pressure exerted by the crystal growing along one side of the boundary (the surface is inclined at an angle of $7^\circ 30'$ with respect to (100)). On the surface of one of the crystals forming an angle of $8^\circ 30'$ with the (100) plane, terrace-shaped figures with a triangular symmetry as shown in Fig. 15 were obtained after very good polishing to a mirror-like finish by means of the finest alundum and after a short (15 min.) etching period in concentrated CP-4 (strong etching agent). Similar figures were obtained by Kikuchi who stressed that it was possible to obtain them only by using a weak agent [5]. After polishing with a coarse grain, irrespective of the etching time (up to two hours) in concentrated CP-4, the terrace-like figures were not produced, and the same surface element had the appearance as shown in Fig. 16. After 30 minutes of rapid etching in concentrated CP-4 and polishing with carborundum with $d \leq 0.075$ mm. there

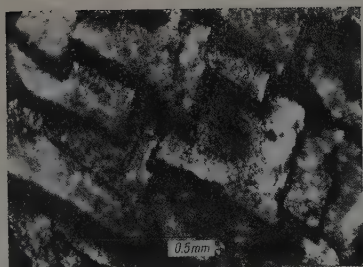


Fig. 1

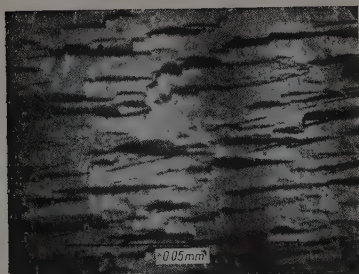


Fig. 2

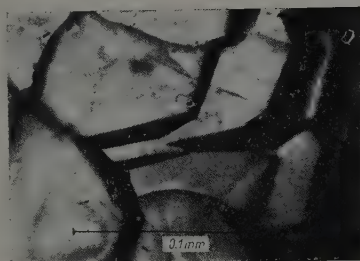


Fig. 3

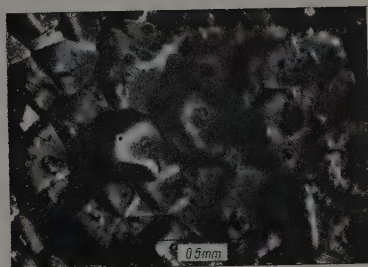


Fig. 4

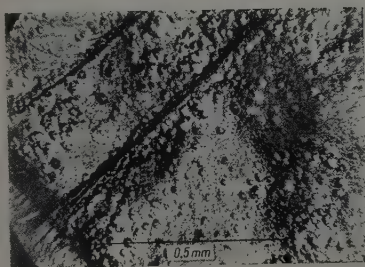


Fig. 5

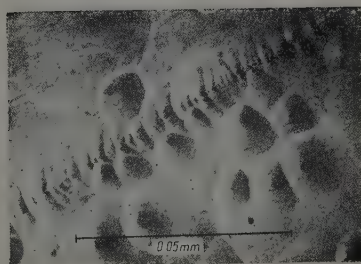


Fig. 6

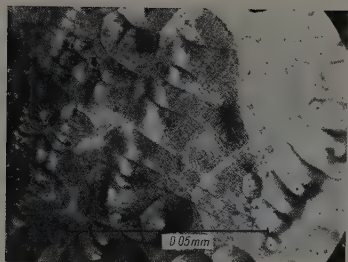


Fig. 7



Fig. 8



Fig. 9

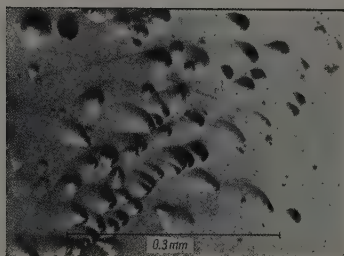


Fig. 10

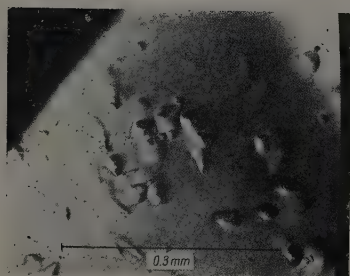


Fig. 11

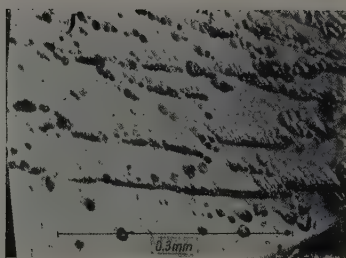


Fig. 12

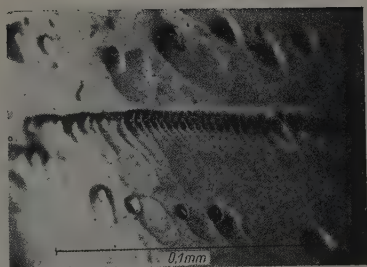


Fig. 13



Fig. 14

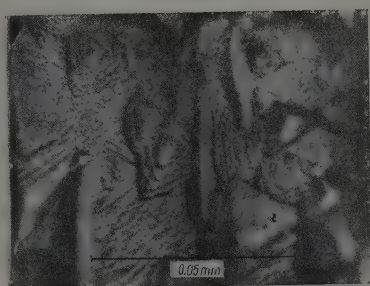


Fig. 15

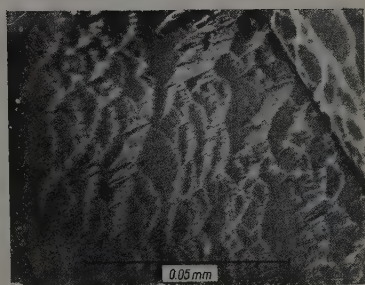


Fig. 16

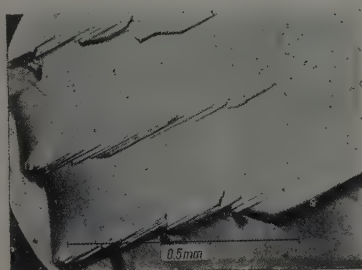


Fig. 17

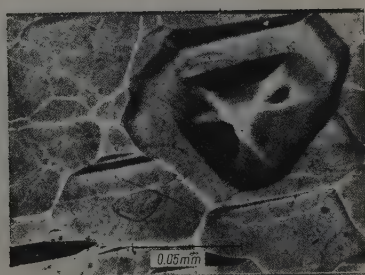


Fig. 18

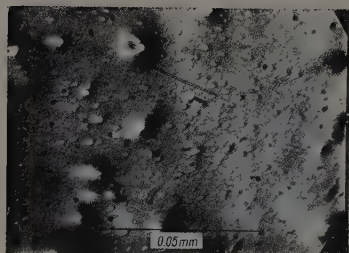


Fig. 19

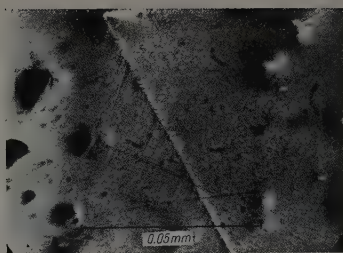


Fig. 20

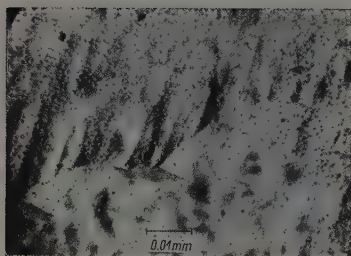


Fig. 21



Fig. 22

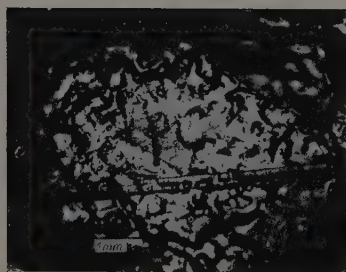


Fig. 23

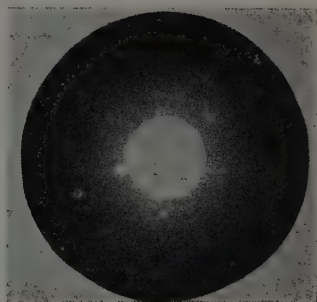


Fig. 24

was observed a different type of boundary consisting of a set of blocks displaced with respect to one another. (Fig. 17 — the surface is inclined at an angle of $7^{\circ}30'$ with respect to (111)).

On one of the crystals grown by the Czochralski technique on a surface making an angle of 5° with (111), the figure shown in Fig. 18 was observed. This figure may have been caused by a screw dislocation.

With very long and slow etching (50 per cent in CP-4 — 50% H_2O) — etching speed of the order of a few hundredths of mm./min. — it was observed that microblock figures appeared (Figs. 19 and 20) on the surfaces of crystals smoothed by two hours of etching in CP-4. The density of the small figures was in this case 10^6 per sq. cm. The surface was at an angle of 11° with (111).

The mirror-like polished surface of a crystal with an orientation of 6° relative to (111) gave, after 30 minutes of etching in CP-4, a figure density of the order of 10^7 per sq. cm. With the increase in the etching time, there was observed an increase in the density of the etched microblock figures as well as large fluctuations in density from 10^6 to 10^7 per sq. cm. (Fig. 21). Fig. 22 shows a surface at 13° with respect to (111). The total etching time is 35 hours in 50 parts of CP-4 + 50 parts of H_2O ; the figure density is 10^8 per sq. cm.

Fig. 23 shows a block ingrown on one of the observed terrace-like figures in Fig. 15. Owing to the use of a fine focus X-ray unit with a focus diameter of $25\ \mu$, it was possible to determine accurately the angle of twist of the blocks. The Laue spots originating from the ingrown block have an elongated shape (Fig. 24). The angle of twist of this block with respect to the neighbouring block is $8^{\circ}45'$.

Conclusions

1. The first stage of the etching corresponding to the observed microblock figures is connected with the removal of the layer destroyed by polishing. It is possible that, in addition to the loosely positioned fragments, the mosaic created during the polishing is also removed from the surface of the blocks, since the etching speed depends on the crystallographic direction. The formation of the microblocks can also be explained by assuming the existence of a number of pits dug into the surface of the crystal coinciding with the cleavage planes.

2. With the increase in etching time, we reach the stage of perfect blocks, and then the crumbling of these leads to the smoothing out of the surface. The time of disappearance of the blocks depends on the orientation of the surface.

3. The conical figures—etch pits—observed most probably correspond to single dislocations. They are identical to those obtained by other authors [2]–[5]. The axes of conical figures correspond to dislocation lines

and their inclination, to the inclination of the dislocation lines with respect to the surface observed. The density of the disclosed dislocations depends on the manner of polishing and the manner and time of etching. It seems that Kurtz, Kulin, and Averbach did not note this fact. The dislocations occur on the boundaries of the microblocks. During the etching, mosaic blocks of decreasing size are removed. After further etching there are observed, at first, conical figures of relatively large size (of the order of $10\ \mu$). These are probably caused by the presence of large local stresses acting on the boundaries of the large mosaic blocks. For this reason the size of these figures is large, the density, small — of the order of 10^5 dislocations/sq. cm. Small stresses would not be revealed, since the etching speed is large at large deformation energies. As a result of still further etching we come to that part of the crystal with still smaller local stresses. The etching surface in the centre of rapid etching should undergo smoothing, since the size of the disturbance is already smaller than the thickness of the layer removed in the course of, say, a few minutes. If we next apply slow etching, all the small local stresses are disclosed and the density of the etched figures increases to 10^9 dislocations/sq.cm.; this was observed by us after sixty hours of etching.

With slow etching the figure density increases with an increase in etching time.

We express our sincere gratitude to Professor L. Sosnowski for his valuable comments during this work and to Mr. Mieluch of Professor Śmiałowski's Physical Chemistry Laboratory as well as to Mr. Brochocki of the Institute of Basic Technical Problems for supplying the germanium monocrystals.

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Intensity and Width of the Raman Lines of Benzene and Carbon Disulphide

by

R. MIERZECKI and N. I. REZAEV

Presented by A. JABŁOŃSKI on April 1, 1957

Recently, theoretical calculations of the intensity of the Raman lines of the benzene molecule were made by Whiffen [1] and of the carbon disulphide molecule, by Volkenstein [2]. Some work has been devoted to experimental measurements of the intensities of the C_6H_6 and CS_2 lines. However, for comparison with the theory it is of interest to investigate the integral intensity of the C_6H_6 and CS_2 lines by a photoelectric method with a narrow setting of the spectrometer slits so that the line contours will not be distorted.

The work was performed on a DFS-4 Soviet diffraction spectrometer by a method described earlier [3], [4]. The Raman spectrum was excited by a low-pressure lamp [4] in a one-lamp elliptical excitation unit. The substance was investigated at room temperature in tubes with a volume of 5 c. cm. The temperature of the substance in excitation unit was 30°C.

The benzene spectrum was studied chiefly by using excitation from the 4358 Å. line of Hg without any optical filter. The 1178 cm^{-1} line was recorded with a filter consisting of a saturated water solution of $NaNO_2$ in order to avoid the superposition of the 2951 cm^{-1} line from the 4047 Hg excitation.

Because of the superposition of the 5026 Å. Hg line on the 3047 cm^{-1} line, the latter was also investigated with 4047 Å. Hg excitation. The investigation showed that the 4047 Å. Hg excitation line broadens the Raman line precisely to the same extent as the 4358 Å. Hg excitation.

The carbon disulphide spectrum was excited by the 4358 excitation line of Hg, the $NaNO_2$ filter being used for protecting the substance from decomposition. Investigated in the carbon disulphide spectrum was

Note: This paper was prepared during R. Mierzecki's leave of absence from the Institute of Physics, Warsaw University.

the line of the fully symmetric vibration with a frequency of 656 cm.^{-1} and the harmonic of the deformation vibration 796 cm.^{-1} enhanced by Fermi resonance with fully symmetric vibration. Other lines were also present in the close vicinity of each of these lines [5], [8], [9]. Thus, all the carbon disulphide lines investigated formed two groups in the spectrograms (Figs. 1 and 2). After obtaining the spectrograms for the lines

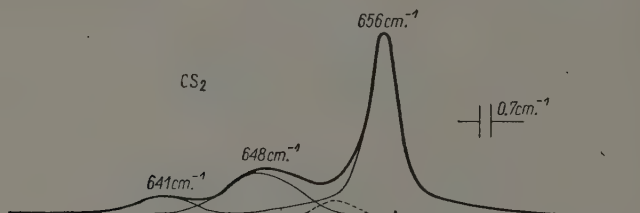


Fig. 1. Group of lines of CS_2 in the 656 cm.^{-1} region

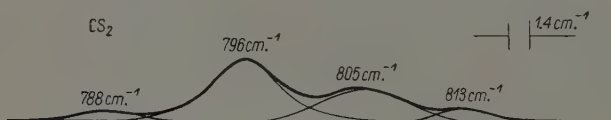


Fig. 2. Group of lines of CS_2 in the 796 cm.^{-1} region

Note: In the figures are shown the observed contours of Raman lines averaged from several spectrograms and the spectral widths of the spectrometer slits.

in each group, the results were analysed by graphical methods, assuming the contours to be symmetric. In the group of $641\text{--}656\text{ cm.}^{-1}$ lines an additional component with a frequency of $\sim 653\text{ cm.}^{-1}$ was obtained which was not observed in other laboratories. Its presence was noted previously in the papers [6], [7] the authors of which used the same equipment. A component was noticed in the position of the spectrum close to the theoretical calculation of the isotopic line [8], [9]. Because of experimental difficulties its intensity was not measured with sufficient accuracy. Difficulties in this spectral region are caused by the presence of false lines produced by the diffraction grating. But their intensity was considerably less than that of the 653 cm.^{-1} component. It is possible that this component is a new Raman line of isotopic origin.

In the measurements of the Raman line intensity the sensitivity of the equipment was taken into account. This was determined by means of the fluorescence spectrum of quinine sulphate [10], [11]. The spectral setting did not change the ratio of the intensities of the polarised and depolarised lines, which were compared by means of a $\lambda/4$ plate.

The Raman lines were investigated by using sufficiently narrow slits, which caused little distortion of their contours. As shown earlier [10],

the ratio of the observed integral intensities of the Raman lines is equal to that of its true integral intensities. For benzene, all investigated line contours were close to the dispersion contours, as seen in [3] and also in Figs. 3 and 4 of the present paper. For carbon disulphide, the contours of some component lines were less distinct because of the weak intensity of the lines.

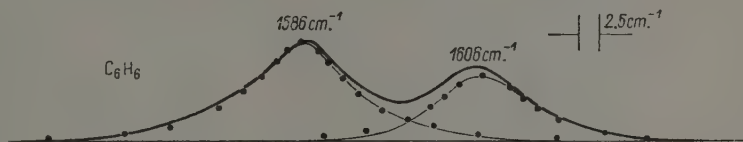


Fig. 3. Group of lines of C_6H_6 in the 1600 cm^{-1} region. The excitation line is 4358 Hg . The points refer to the theoretical curves of the dispersion contour

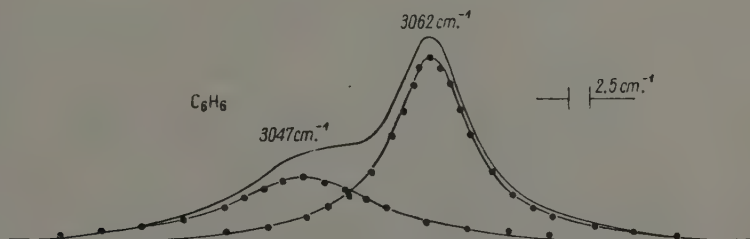


Fig. 4. C_6H_6 lines in the 3060 cm^{-1} region. The excitation line is 4047 Hg . The points refer to the theoretical curves of the dispersion contour

The width of the lines was found by a known method. The observed contours were corrected for the distortion caused by the slits by the graphical method of Rayleigh (Strutt) [12]. The observed width of the excitation line together with the instrumental function [11], [13] was subtracted from the observed width of the Raman line.

The line widths δ measured in this way, the integral intensities I_∞ , and the observed intensities at the maximum I_0 are shown in Table I. In the case of benzene, for each line about 10 measurements were made. The mean error of measurement of the width of the lines and the intensity at the maximum was near 5% and the integral intensity, 10%. The carbon disulphide lines were measured 6–7 times. For the most intensive lines, the mean error of measurement was 10% and for weak lines, it increased accordingly. All the intensities are expressed on a scale based on the 802 cm^{-1} line of cyclohexane for which I_0 equals 250 units and I_∞ equals 500 units [11]. Comparison of the intensities of the lines of benzene indicated that the values of I_∞ measured in the present work are in sufficiently good agreement with the data of photoelectric mea-

surements of other authors [11], [14], [15]; the experimental values of the intensities of the carbon disulphide lines found by different authors differ somewhat (Table II). Evidently, this is connected with

TABLE I

Experimental values of the intensities and widths of Raman lines

C_6H_6 molecule				CS_2 molecule			
Freq. cm^{-1}	I_0	I_{∞}	δ cm^{-1}	Freq. cm^{-1}	I_0	I_{∞}	δ cm^{-1}
607	67	330	7.1	641	150	300	—
850	28	140	8.1	648	430	1000	3.7
992	1100	2200	1.8	656	2100	3300	0.9
1178	60	390	10.8	788	30	100	—
1586	46	360	11.6	796	200	670	5.2
1606	32	230	10.5	805	100	330	5.5
2951	43	140	3.1	813	40	130	—
3047	140	1200	14.8				
3062	360	1770	7.0				

TABLE II

Comparison of theoretical and experimental intensity values

C_6H_6 molecule

Freq. cm^{-1}	Symmetry species	Whiffen [1] (theor.)	Rank [14]	Langsberg [11]	Allen [15]	Present work
607	E_{2g}	2.1	1.3	1.4	1.3	1.6
850	E_{1g}	1.04	0.47	0.40	0.52	0.73
992	A_{1g}	12.0	12.0	12.0	12.0	12.0
1178	E_{2g}	2.9	2.2	1.9	2.1	2.2
1586}	E_{2g}	3.8	2.9	3.3	2.9	3.6
1606}						
3047	E_{2g}	8.2	8.1	—	6.5	9.8
3062	A_{1g}	17.8	13.4	—	15.7	14.5

CS_2 molecule

Freq. cm^{-1}	Volkenstein [2] (theor.)	Pieńkowski [16]	Langseth *) [17]	Caldirola *) [8], [9]	Rank [14]	Present work
641 isotop.	0.9	—	—	0.9	—	0.9
648 excit	2.3	—	2.2	2.9	4.4	3.1
656	10.0	10.0	10.0	10.0	10.0	10.0
788 isotop.	0.14	—	0.05	0.17	—	0.30
796	1.6	3.2	0.79	1.75	1.5	2.1
805 excit	1.3	—	0.26	0.9	0.9	1.0
813 excit	—	—	0.05	0.2	—	0.4

*) The line intensities were measured at the maxima. The authors give the ratio of intensities measured according to the area under the contours only for the 656 and 796 lines. This ratio is 10:3.3 in [17] and 10:1.9 in [8], [9].

the difficulties in analysing closely situated components of fine structure and with differences of the methods used by different authors [8], [9], [14], [16], [17] measuring the integral intensity.

Theoretical calculations of the intensity of the benzene lines [1] are shown in Table II. Given in this table are the experimental values of the integral intensities as measured in the present work, as well as the results of other authors [11], [14], [15] obtained by photoelectric methods. For comparison with theory [1] all experimental values of the integral intensities are multiplied by $\left(1 - \frac{\nu}{\nu_0}\right)^{-4} (1 - e^{-h\nu/kT})$ and expressed on a scale for which the intensity of the 992 cm^{-1} line equals 12.0. Here ν is the frequency of the Raman line, ν_0 is the frequency of the excitation line, T is the absolute temperature. As may be seen from Table II, the experimental values of the present work are in agreement with the theoretical calculations. The greatest difference amounts to 20–30%. The 2951 cm^{-1} line is a harmonic of the frequency 1480 cm^{-1} and its intensity was not calculated theoretically.

The intensity of the oscillation spectrum of carbon disulphide computed from the Fermi resonance calculations and the isotopic structure [2] are given in Table II, in which are given, in addition to the experimental values of the integral intensities of the present work, also data obtained by other authors [8], [9], [14], [16], [17]. The intensities of all lines are expressed on a scale for which the intensity of the 656 cm^{-1} line is assumed to be 10.0. The experimental values obtained in the present paper for carbon disulphide are in good agreement with the theoretical calculations, although for some lines the difference is perceptible.

In conclusion, we wish to thank Professor P. A. Bazhulin for his constant interest and valuable advice in the course of the investigations.

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On Preparation of N, 2, 4, 6-Tetranitromethylaniline by an Action of Nitric Acid on Dimethylaniline

by

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Presented by T. URBANŃSKI on March 13, 1957

N, 2, 4, 6-tetranitromethylaniline (Tetryl) is usually obtained by nitrating dimethylaniline sulphate dissolved in an excess of sulphuric acid. Nitration is carried out by means of a nitrating mixture, rich in nitric acid.

A few attempts at nitrating dimethylaniline by means of nitric acid alone were, generally speaking, unsuccessful, and this led to a deeply rooted conviction that formation of dimethylaniline sulphate prior to its nitration is an essential condition to a safe nitration.

However, Hodgson and Turner [1] have found that dimethylaniline can be nitrated on a small scale (samples of 5 g. of dimethylaniline) with an excess (100 ml.) of nitric acid. When they used nitric acid of d 1.52 at -5° to 0°C ., tetryl resulted. When concentration of nitric acid was lower (e. g., 1.42) the reaction did not lead to the product of full nitration but only N, 2, 4-trinitromethylaniline [2] (which Hodgson and Turner thought to be 2, 4, 6-trinitromethylaniline) resulted.

According to Hodgson and Turner, nitric acid d 1.34 and 1.254 yielded 2, 4-dinitrodimehtylaniline or (on prolonged reaction time) 2, 4-dinitromethylaniline.

Lang [3] pointed out, that N-nitroderivatives cannot be obtained by acting on 2, 4-dinitro- or 2, 4, 6-trinitro-N-methylaniline with nitric acid of concentration lower than 70%.

The authors of the present paper have found that tetryl can safely be prepared by acting with a great excess of nitric acid d 1.40 on dimethylaniline. The essential condition consists in carrying out the reaction in two stages. In the first stage dimethylaniline is dissolved in nitric acid d 1.40 at a temperature which does not exceed 7°C . After keeping the solution for some time at a low temperature, it is subjected to careful warming. In this second stage of the reaction, the temperature should not exceed 80°C . and only at the end of this stage, can the temperature be raised to 90°C . before the solution is cooled.

It is dangerous to dissolve dimethylaniline in nitric acid of high concentration (d 1.51) as the reaction may become too violent, but it is advisable to add nitric acid of d 1.51 to the already prepared solution of dimethylaniline in nitric acid d 1.40 and after that to warm it up. This procedure increases the yield of the product and its purity.

Another essential condition consists in using a great excess of nitric acid; the $\frac{\text{nitric acid}}{\text{dimethylaniline}}$ ratio should be kept near to 40 or even higher.

When the ratio is 35 or lower, the purity of tetryl is unsatisfactory.

Vigorous mixing is also important for the safety of the reaction.

The purity of tetryl obtained with nitric acid alone is higher than that of tetryl prepared by the usual method, in presence of sulphuric acid. It should be pointed out that sulphuric acid can denitrate the N-nitro group [4], [3]. On the other hand, a high content of sulphuric acid in the nitrating mixture may promote formation of a higher nitrated product — N,2,3,4,6-pentanitromethylaniline [5].

These facts may explain the high purity of the product obtained by the method described in the present paper.

Experimental

Nitration with nitric acid d 1.40. 6 g. of dimethylaniline were dissolved in 280 g. nitric acid (d 1.40), keeping the temperature at 2–7°C. and mixing vigorously. The solution was coloured green due to the presence of nitrosocompounds formed in the early stage of the reaction. After all the dimethylaniline had been dissolved, the solution was kept for c. 15 min. at 2°C. and was then carefully warmed on a water bath to 40–50°C. At this stage a considerable evolution of heat and gases — mainly nitrogen oxides occurred, and heating was stopped for c. 30 min.

After the reaction subsided, the dark reddish solution was warmed again to 80°C. After a certain time, brown fumes of nitrogen oxides ceased being evolved and the solution changed to a light orange colour.

To secure a complete reaction the temperature was, at this stage, raised to 90° and kept at this temperature for one hour.

The solution was cooled down to 10°C.; 50 ml. water were added and after a few hours of standing at room temperature, the crystalline product was collected, washed with cold and then with hot water. The yield of tetryl was 10.6 g. (78 per cent of theoretical), m. p. 127–128°.

Nitration with mixtures of nitric acid d 1.40 and 1.51. 6 g. of dimethylaniline were dissolved (as in the previous method) in 84 g. of nitric acid (d 1.40). After that, 150 g. of nitric acid (d 1.51) were added, the temperature still being kept between 4 and 5°C. The reaction solution was next warmed as in the previous method. After the reaction was terminated, the solution was cooled and 100 ml. water were added.

Tetryl crystallised out. It was filtered, washed with cold and then with hot water. The yield was 11.3 g. (83.1 per cent of theoretical) of a product of high purity, m. p. 129.5°C.

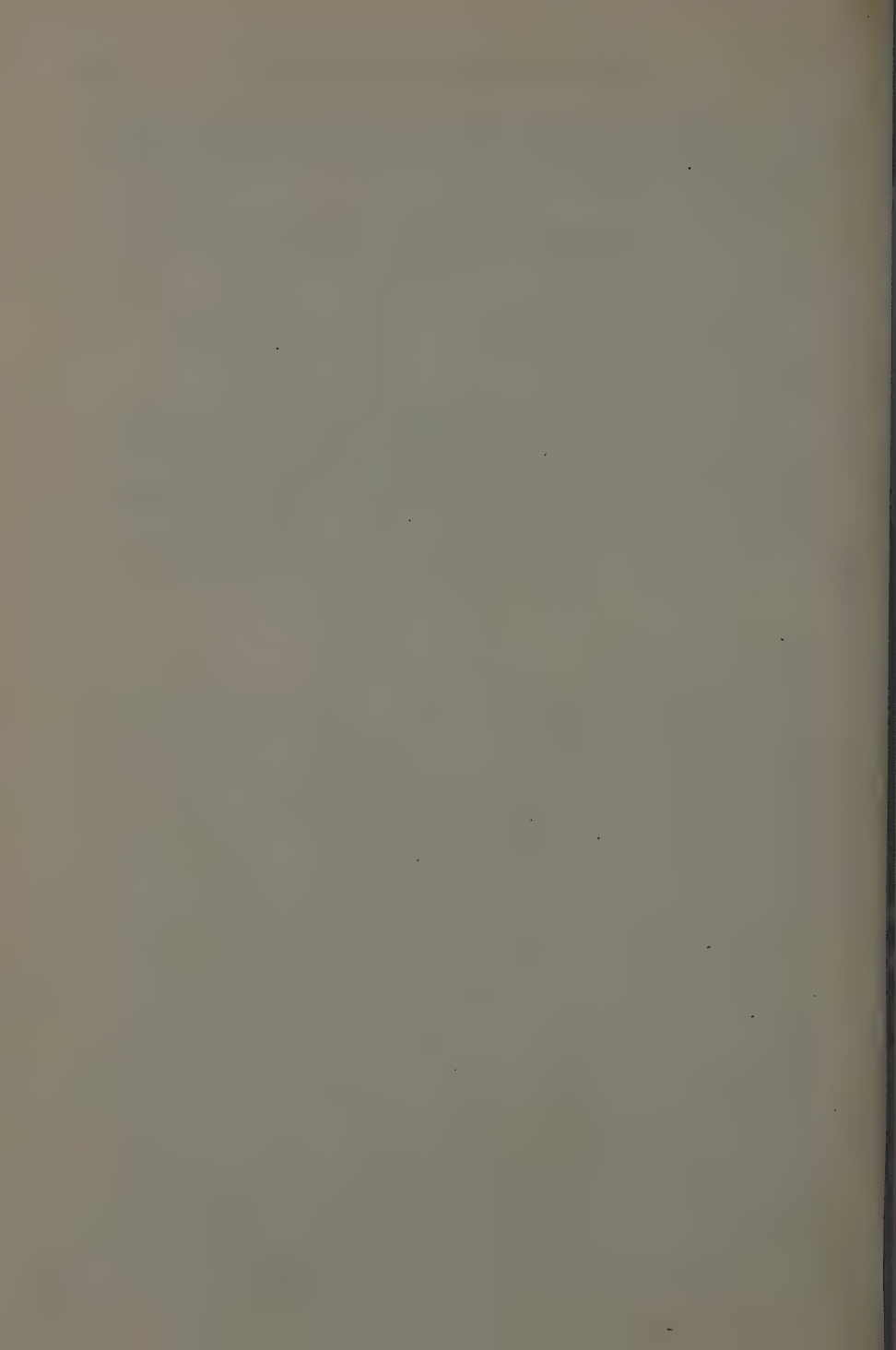
Analysis:

calculated for $C_7H_5O_8N_5$	24.4 % N,
found	24.8 % N.

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On Ultraviolet Absorption Spectra of Some β -Halogen Derivatives of β -Nitroalcohols. II

by

W. SOBÓTKA, Z. ECKSTEIN, T. URBĄŃSKI

Presented by T. URBĄŃSKI on March 18, 1957

In one of our previous papers we examined the ultraviolet absorption spectra of some β -halogen derivatives of β -nitroalcohols [1]. It was then found that the shape of the absorption curve depends on the halogen attached to carbon, combined with the nitro group. The secondary β -nitroalcohols have shown absorption maxima near to $275\text{ m}\mu$. The intensity of the maxima was reduced by substitution with chlorine of the active hydrogen. The introduction of bromine changed the maximum into a bend.

To continue this line of research 1-nitromethyl-1-hydroxycyclohexane (I) and its chloride (II) and bromine (III) derivatives of the general formula (A) have been prepared.

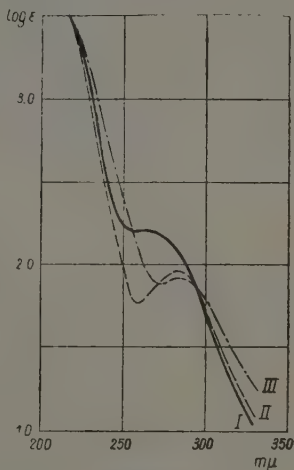
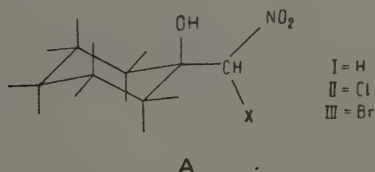
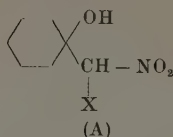


Fig. 1

The physical constants of the compounds have now been checked and completed, and the molecular refractions calculated. Acetic acid esters (IV–VI) of the compounds (I–III) have also been prepared. All data concerning the compounds were tabulated (Table I).

The ultraviolet absorption curves are shown in Fig. 1.

TABLE I

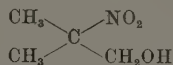


No.	R	X	b. p. (°C.)	at mm. Hg.	n_D^{20}	d_{20}^{20}	R_{MD}		Reference
							calc.	found	
I	H—	H—	93	0.4	1.4881	1.1616	40.02	39.47	[3]-[9]
II	H—	Cl—	96.5	0.4	1.5000	1.2907	44.80	44.09	[10]-[11]
III	H—	Br—	118	1	1.5208	1.5476	47.70	46.80	[10]-[11]
IV	CH ₃ CO—	H—	116	1	1.4669	1.1468	49.27	48.43	[10]-[11]
V	CH ₃ CO—	Cl—	32.5 *)	—	—	—	—	—	
VI	CH ₃ CO—	Br—	125	1	1.4950	1.4500	56.94	56.05	

*) m. p.

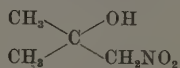
Only a very slight maximum ($\Delta\epsilon=6$) due to the nitro group characterises the unsubstituted compound (I). The substitution with chlorine (compound II) considerably increases the intensity of the maximum ($\Delta\epsilon=37$) and is shifting it towards longer waves. Bromoderivative (III) gives a maximum stronger ($\Delta\epsilon=10$) than the unsubstituted compound, but weaker than the chloroderivative (II).

The surprising weakness of the NO₂-band ($\Delta\epsilon=6$) in the compound (I) led to a suggestion that this might be due to the fact that the compound (I) was a tertiary alcohol. Formerly, in the secondary β -nitroalcohols mentioned in our previous paper, $\Delta\epsilon$ was 9.4–9.6 [1] and $\Delta\epsilon=12.2$ was found in the case of the structure



(compound XIV, [3]).

In order to verify this suggestion the ultraviolet absorption spectrum of the tertiary β -nitroalcohol (VII), i. e., 2-nitromethyl-2-hydroxypropane has been examined.



(VII)

It was prepared by the known method of aldol condensation of acetone with nitromethane [9]. After purification of the product by repeated distillation under reduced pressure it showed physical constants as follows: $n_D^{20}=1.4430$ *); $d_{20}^{20}=1.1313$ *); b. p. 55.5–56.5°C./1.3 mm. Hg.; m. p. 22°C. The calculated value of molar refraction was 27.91, and the

*) All values for supercooled liquid.

experimental value -27.89^*). It is worth pointing out, that the literature [9] described the compound as liquid and no m. p. was given.

The absorption curve of (VII) is given in Fig. 2. There is a strong maximum of the NO_2 group ($\Delta\epsilon=26$) between 275 and 280 $m\mu$.

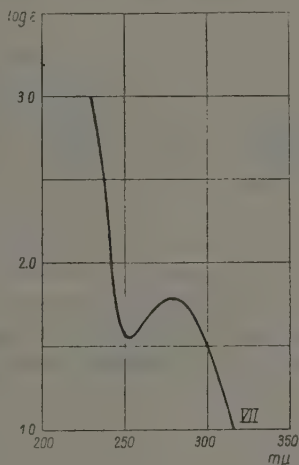


Fig. 2

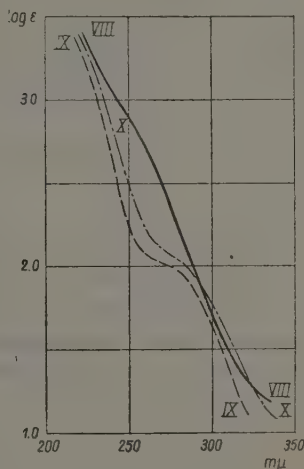
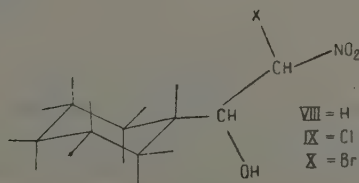


Fig. 3

Another suggestion was therefore put forward concerning the influence of the cyclohexane ring on the nitro group. To verify this suggestion a new series of compounds — 2-nitro-1-cyclohexylethanol (VIII) and its chloro- and bromo-derivatives (IX) and (X) — have been prepared. They are higher homologues of compounds (I), (II) and (III), respectively. The position of the nitro group to the cyclohexane ring is different in the two series of compounds, (I–III) and (VIII–X), as shown in the discussion below.

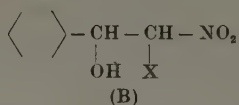


B

The product (VIII) was obtained by aldolic addition of cyclohexyl-formaldehyde to nitromethane. The chlorination and bromination of sodium salts of nitroalcohol (VIII) led to the compounds (IX) and (X) respectively. The properties of these compounds are given in Table II and their absorption spectra in Fig. 3.

*) All values for supercooled liquid,

TABLE II



No.	X	b. p. (°C.)	at mm. Hg.	yield (%)	n_D^{20}	d_{20}^{20}	R_{MD}	
							calc.	found.
VIII	H—	123	1	88.5	1.4880	1.1194	44.64	44.52
IX	Cl—	118	1.2	54.0	1.4952	1.2365	49.41	48.83
X	Br—	132	1.5	48.5	1.5160	1.4764	52.32	51.55

The unsubstituted product (VIII) does not show the maximum of the nitro group. The band near 270–275 $m\mu$ appears in the case of the chloro- and bromoderivatives, (IX) and (X), respectively. The effect of substitution of both the nitroalcohols (I) and (VIII) is very similar. The characteristics of the absorption spectra are tabulated below (Table III).

TABLE III

Characteristics of the absorption spectra (in ethanolic solution)

No.	λ min.	ϵ min.	λ max.	ϵ max.	$\Delta\epsilon$
I	256	156	265–270	162	6
II	260	58	285	95	37
III	272	80	285	90	10
VII	255	36	280	62	26
VIII	—	—	—	—	—
IX	—	—	268–270 *)	109	—
X	—	—	270–275 *)	119	—

*) a bend.

Discussion

The action of halogen substitution in 1-nitromethyl-1-hydroxycyclohexane (I) and 2-nitro-1-cyclohexylethanol (VIII) upon the ultraviolet absorption band near 270 $m\mu$ is the reverse of the action of halogen substitution in the aliphatic straight chain β -nitroalcohols previously described [1].

The weakness and the lack of NO_2 -maximum in the absorption curves of the compounds (I) and (VIII), respectively, could not be explained only by the existence of a hydrogen bond between the OH and the NO_2 groups. According to our former observation [2], [3] two OH groups are required to suppress the maximum produced by a nitro group or to transform it into a shoulder. This, however, was found, where

simple alkyl radicals and no rigid ring system were present. In the instance of cyclohexane derivatives, the stiff ring can produce a steric effect sufficient to reduce the intensity of the absorption band of the nitro group. Together with the hydrogen bond this may result in suppressing the NO_2 -maximum, as is the case of the compound (VIII).

Introduction of chlorine or bromine into the molecule (compounds (II), (III), (IX), (X)) may weaken the hydrogen bond between the NO_2 and OH groups and would subsequently have an opposite effect to that of a cyclohexane ring.

The authors are much indebted to Professor D. H. R. Barton for a useful discussion of the results.

Experimental

1-Nitromethyl-1-acetoxycyclohexane (IV). The mixture of the compound (I) (16 g., 0.1 mol.), acetyl chloride (15.6 g., 0.2 mol.) in anhydrous chloroform (30 ml.) was boiled for 12 hours under a reflux condenser. After removing the solvent and unreacted acetyl chloride the residue was distilled under reduced pressure; 13 g. of the product were collected at $115.5\text{--}116^\circ/1$ mm. Hg.

1-(Nitrochloromethyl)-1-acetoxycyclohexane (V). The mixture of compound (II) (19.3 g., 0.1 mol.) and acetyl chloride (15.6 g., 0.2 mol.) in anhydrous chloroform (30 ml.) was boiled for 4 hours under a reflux condenser. After removing the solvent and unreacted acetyl chloride, 19.5 g. of the product were collected at $113\text{--}117^\circ/1$ mm. Hg. The product melted at $31.5\text{--}32.5^\circ$. (Analysis: found N 6.1%; $\text{C}_9\text{H}_{14}\text{O}_4\text{NCl}$ requires N 5.9%).

1-(Nitrobromomethyl)-1-acetoxycyclohexane (VI). Acetyl chloride (15.6 g., 0.2 mol.) was added to the mixture of compound (III) (23.8 g., 0.1 mol.) in anhydrous chloroform (30 ml.) and boiled 4 hours under a reflux condenser. After removing the solvent and acetyl chloride, 6.7 g. of the product were collected at $125\text{--}125^\circ/1$ mm. Hg. (Analysis: found N 5.0%; $\text{C}_9\text{H}_{14}\text{O}_4\text{NBr}$ requires N 4.9%).

2-Nitro-1-cyclohexyl-ethanol-1 (VIII). Cyclohexylformaldehyde (112 g., 1 mol.) was mixed with nitromethane (70 g., 1.15 mol.), water (5 ml.) and triethylamine (10 ml.) and left for 48 hours at 35°C . Ether was then added, and the whole neutralised with hydrochloric acid. The ethereal extract was separated, washed with aqueous sodium-hydrogen carbonate and dried over sodium sulphate. The ether was evaporated and product (VIII) was distilled at $123^\circ/1$ mm. Hg (bath temperature 160°). (Analysis: found N 8.1%; $\text{C}_8\text{H}_{15}\text{O}_3\text{N}$ requires N 8.1%).

2-Nitro-2-chloro-1-cyclohexylethanol-1 (IX). Compound (VIII) (43 g., 0.25 mol.) was dissolved in anhydrous methanol (80 ml.). A solution of sodium (5.7 g., 0.25 mol.) in methanol (80 ml.) was added

at a temperature of 0°. The sodium salt of (VIII) precipitated, was filtered off and washed with ether. The yield was 42 g.

The sodium salt of (VIII) was suspended in anhydrous chloroform (100 ml.) and chlorine was introduced at 0°. The end of chlorination was determined by igniting samples of suspension until no organic substance was present in the solid. Sodium chloride was filtered off, the solvent was evaporated, and product (IX) was distilled under reduced pressure at 118–119°/1 mm. Hg (bath temperature 145°). (Analysis: found N 6.6%; $C_8H_{14}O_3NCl$ requires N 6.7%).¹

2-Nitro-2-bromo-1-cyclohexyl-ethanol-1 (X). The sodium salt of (VIII) was brominated in chloroform (as above). Colourless oily product (X), b. p. 132°/1.5 mm. Hg., resulted. (Analysis: found N 5.4%; $C_8H_{14}O_3NBr$ requires N 5.5%).

The authors are much indebted to Mrs. J. Świątosławska, Miss D. Ciecierska and Miss K. Gorczyńska of the Institute of General Chemistry in Warsaw for carrying out experiments on absorption spectra by means of a Beckman spectrophotometer.

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Decomposition of Potassium Ferrate (VI) in Concentrated KOH Solutions

by

B. JEŻOWSKA-TRZEBIATOWSKA and M. WROŃSKA

Presented by W. TRZEBIATOWSKI on March 18, 1957

Our earlier investigations showed that the manganese oxyanions MnO_4^- , MnO_4^{2-} and MnO_4^{3-} are reduced by concentrated alkaline solutions. The kinetics and mechanism of these reactions were investigated [1]–[3]. We have now extended these investigations to the other oxyanions of the transition elements beginning with the ferrate (VI).

It is known that potassium ferrate decomposes in aqueous solution into Fe (III) and oxygen and that their stability increases with increasing OH^- ion concentration [4], [5].

We investigated the concentrated and dilute solutions of ferrate (VI) and established that the latter, in contrast to the former, decomposed more rapidly when the concentration of alkali increased. This finding shows that the mechanism of decomposition also varies with the ferrate (VI) concentration. In the present paper we report the results of our measurements in very dilute solutions of potassium ferrate (VI).

The reaction was carried out in presence of a very considerable excess of OH^- ions, which offered a constant ionic strength to the solution and enabled this reaction to be regarded as pseudo-unimolecular. The progress of the reaction was determined oxydometrically [6], at temperatures constant within 0.1° . Potassium ferrate (VI) was obtained according to the method of Schreyer, Thompson and Ockerman [7]; water was double distilled with a content of KMnO_4 and the potassium hydroxide was of p. a. grade and was additionally controlled by spectrochemical analysis.

Rate measurements were carried out with potassium hydroxide concentrations of 7 and 10 moles per litre and at 20 and 30° . The concentration of potassium ferrate varied in the ranges of 0.0025 – 0.00065 M.

The reduction curves represent the actual concentrations of potassium ferrate (VI) in terms of oxygen equivalents. In all experiments

the rate of the reduction of K_2FeO_4 was at first proportional to the concentration of potassium ferrate (VI); this indicated that the primary reaction obeyed the first order law. The rate of the reaction can be represented by the expression:

$$-\frac{d[FeO_4^{2-}]}{dt} = k(a-x),$$

where a is the initial concentration of K_2FeO_4 , x the amount of product at time t .

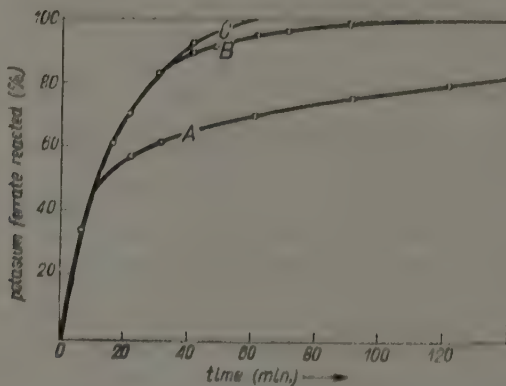


Fig. 1. Rate of reduction of potassium ferrate (VI) at various initial concentrations: A = 0.0025, B = 0.00125 and C = 0.00065 M in 7 M potassium hydroxide solution at a temperature of $+20^\circ$

TABLE I
First order kinetics

temp. $+20^\circ$ 7 M KOH			temp. $+30^\circ$ 7 M KOH			temp. $+30^\circ$ 10 M KOH		
t min.	react. %	k	t min.	react. %	k	t min.	react. %	k
10	45	0.059	10	63	0.090	5	63	0.108
15	60	0.061	15	77	0.098	10	86	0.106
20	70	0.060	20	86	0.098	15	95	0.100
30	80	0.061	30	95	0.103	20	98	0.108

The rate of the reduction of K_2FeO_4 was proportional to the concentration of potassium hydroxide and decreased with its diminution (Fig. 2).

During the progress of the reduction the reaction order diminished. The rate of production of Fe (III) is given by $\frac{d[Fe(III)]}{dt} = k \frac{a-x}{a-x}$ in

agreement with the half order kinetics observed with respect to oxygen equivalents.

The change of the order of the reaction always occurred at the same concentration of iron (III) after reaching the solubility product of $\text{Fe}(\text{OH})_3$, which was confirmed analytically ($\text{L}[\text{OH}^{++}]^3 = [\text{FeO}_4^{2-}] = 1.12 \cdot 10^{-23}$ at $20^\circ\text{C}.$)

It is probable that the precipitation of iron hydroxide initiated the reaction occurring on the surface of $\text{Fe}(\text{OH})_3$ which diminished the speed of the reduction.

At very low initial concentration of potassium ferrate (VI) (0.00065 M) at $+20^\circ$ and at higher concentration at $+30^\circ$ (Fig. 2) the solubility pro-

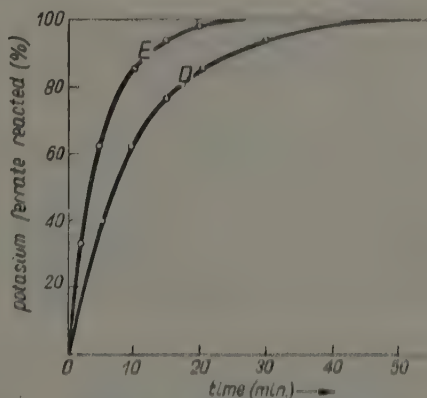


Fig. 2. Rate of reduction of 0.00125 M K_2FeO_4 at various concentrations of potassium hydroxide: D — 7 and E — 10 M at $+30^\circ$

TABLE II

One half order kinetics

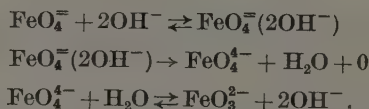
temp. = $+20^\circ$ $a = 0.0014$ M K_2FeO_4			temp. = $+20^\circ$ $a = 0.00015$ M K_2FeO_4		
t min.	react. %	$10^3 k$	t min.	react. %	$10^3 k$
21	30	0.207	14	41	0.200
51	45	0.203	19	49	0.204
81	56	0.198	34	75	0.208
111	63	0.200	54	84	0.208

duct of ferric hydroxide was not reached and therefore the experimental data for the overall reaction followed the first order kinetics.

The energy of activation calculated from the rate coefficients for the first order progress of the reaction is 8.6 kcal.

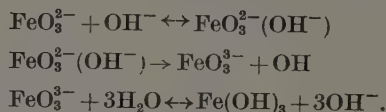
The results observed can be explained by a mechanism, which involves the intermediate formation of ferrate (IV). The appearance of

this oxidation state was confirmed oxidimetrically in the experiments with concentrated potassium ferrate (VI) in concentrated KOH solutions. The black ferrate (IV) obtained in these conditions was precipitated on the surface of ferric hydroxide. The first step of the reduction is therefore believed to be:



The co-ordination of two OH^- ions is in agreement with the low value of activation energy.

The FeO_3^{2-} ions produced in the first step may react with OH^- ions to give FeO_3^{3-} :



It should be mentioned that the rate of the reduction of K_2FeO_4 in concentrated KOH solutions was accelerated by a content of Rb^+ ions or inhibited by Na^+ ions. This finding will be discussed in a later paper.

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Disproportionation of Sodium Metabisulphite ($\text{Na}_2\text{S}_2\text{O}_5$) in the Presence of Sodium Hydroxide (NaOH)

by

R. RUDNICKI

Presented by T. MIŁOBĘDZKI on March 23, 1957

Continuing the investigations on reactions in solid state [2]–[10] carried out under the direction of T. Miłobędzki, and subsequently the studies on disproportionation of pure sodium metabisulphite [1], the influence of sodium hydroxide on the disproportionation of sodium metabisulphite was examined.

Experimental

All the experiments were carried out according to the method described in [1].

In the experiments were used $\text{Na}_2\text{S}_2\text{O}_5$ (composition given in [1]) and sodium hydroxide (E. K. A., Sweden) of the following composition:

NaOH — 92.9 %

Na_2CO_3 — 4.3 %

H_2O — 2.8 % (calculated from the difference).

The nitrogen, in the atmosphere of which the experiments were performed, was supplied from a steel cylinder. Oxygen was removed by a glowing copper gauze. Oxygen-free nitrogen, dried over calcium chloride, was passed through a wash bottle with concentrated sulphuric acid, for controlling the velocity of flow, and then through a reaction tube placed in an oven. Nitrogen with reaction gases was passed through a wash bottle containing a solution of iodine in potassium iodide (absorption and oxidation of SO_2 to SO_3), through a wash bottle with a solution of potassium iodide (absorption of iodine vapours entrained from the preceding wash bottle), then through U-tubes with calcium chloride and over glowing CuO , on which hydrogen (if present) was burnt to water, the latter being absorbed on calcium chloride.

A similar apparatus was used for the qualitative determination of water liberated in the reaction.

The reaction of decomposition of pure sodium metabisulphite and of sodium metabisulphite in the presence of sodium hydroxide were carried out in a porcelain boat placed in a Jena-glass tube.

In the experiments c. 1.0000 g. of $\text{Na}_2\text{S}_2\text{O}_5$ and 0.8416 g. of NaOH were used, i. e., quantities calculated from the point of view of the possibility of complete oxidation of S^{+4} to S^{+6} by protons obtained from sodium hydroxide.

The quantity of ground NaOH required was weighed in a weighing bottle, quickly mixed with a weighed quantity of sodium metabisulphite and put into the porcelain boat. The sodium metabisulphite residue left on the walls of the weighing bottle was transferred to a solution containing an excess of iodine, and the latter was titrated with hyposulphite. This correction was taken into account in the calculations.

The results obtained are presented in Table I and in diagram 1.

TABLE I
Heating of $\text{Na}_2\text{S}_2\text{O}_5$ with 4 NaOH

Temperature °C.	Reaction time hours	S^{-2}	S^0	S^{+4} Na_2SO_3	S^{+4} SO_2	S^{+6}	Quantity of water from the combustion of hydrogen
		per cent of sulphur S^{+4} quantity undergoing disproportionation					
100	4	—	present	89.20	—	2.42	—
200	4	—	present	89.76	0.33	7.74	—
300	4	—	present	18.72	0.34	48.21	—
400	4	traces	present	23.21	0.32	47.85	—

In addition to the series of experiments which established that in the reaction conditions hydrogen is not liberated, i. e., oxidation does not take place [11], further investigations on the quantity of water liberated in the reaction were carried out.

As in the previous experiments 0.8416 g. of ground NaOH and 1.0000 g. of $\text{Na}_2\text{S}_2\text{O}_5$ were weighed in a weighing bottle. Both substances were mixed and put in a porcelain boat.

The liberated water was determined from the difference of weights of the U-tube filled with calcium chloride.

The results obtained are presented in Table II and in diagram 2.

Water was liberated following the reaction



The quantities of SO_2 , leaking out and thus not taking part in the reaction, were minute (see Table I).

TABLE II

Quantity of water evolved (per cent) during the heating of $\text{Na}_2\text{S}_2\text{O}_5$ with 4NaOH in a porcelain boat

Temperature °C.	Time of heating hours	Quantity of water evolved %
100	1	10.93
100	1	12.30
100	1	9.16
100	1	7.91
200	1	14.03
200	1	15.77
300	1	15.87
300	1	21.37
300	1	17.37
400	1	20.84
400	1	16.45
200	4	25.00
300	4	24.80
400	4	25.50

Anhydrous NaOH contains 22.52% of H_2O ($\text{Na}_2\text{O} + \text{H}_2\text{O} = 2\text{NaOH}$). However, in the process of grinding, sodium hydroxide absorbs moisture

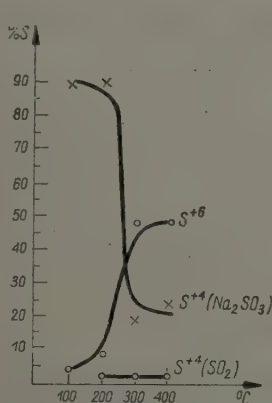


Fig. 1. Heating of $\text{Na}_2\text{S}_2\text{O}_5$ with 4NaOH for 4 hours in nitrogen atmosphere

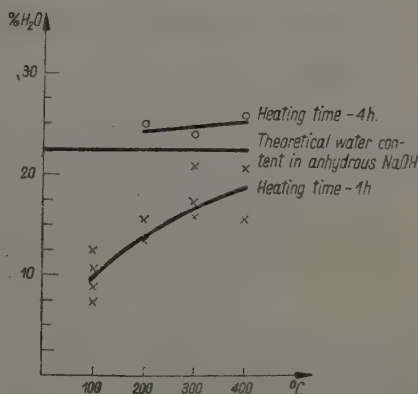


Fig. 2. Amount of water liberated during heating of $\text{Na}_2\text{S}_2\text{O}_5$ with 4NaOH in porcelain boat in nitrogen atmosphere

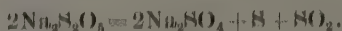
from the air, hence in the experiments here described, more than 22.52% per cent of water was liberated, since the water present in the initial preparation (see composition of NaOH) and the water absorbed during grinding was also evolved.

Discussion of results

Sodium hydroxide has the following influence on the disproportionation of sodium metabisulphite: in the temperature range of between 100–200°C. a reaction similar to that of disproportionation of pure $\text{Na}_2\text{S}_2\text{O}_5$ [1] took place with the only difference that NaOH combined with the sulphurous anhydride evolved.

Sulphur dioxide, which was liberated in the experiments only in very small quantities, was derived from the decomposition of metabisulphite molecules present in the surface layer of the mixture.

It was found that from 300°C. up, the reaction had to run as follows:



SO_2 reacted with NaOH forming sodium sulphite and water. This sulphite was disproportionated at 100°C., yielding sulphide and sulphate. This was determined by the presence of sulphides in the products of disproportionation of metabisulphite at 400°C.

Summary

The disproportionation of sodium metabisulphite in the presence of sodium hydroxide in the temperature range of 100–400°C. was investigated and the quantities of water liberated in the reaction were determined.

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The Critical Region of Liquids. II. The Limited Miscibility of Compressed Gases and the Influence of Impurities on Hysteresis Phenomena

by

A. KRĘGLEWSKI

Presented by W. ŚWIEȚOSŁAWSKI on April 4, 1957

Introduction

In a previous paper [2] we discussed the influence of gravity and of mixing on the liquid-vapour coexistence curve in the critical region of pure substances and mixtures. As to the phenomena taking place outside the coexistence curve, in the region at and above the temperature of disappearance of the meniscus T_m , hysteresis phenomena seem to be most characteristic. They were first observed and extensively investigated by Maass and co-workers. These investigations were continued by the author and have been described in a number of papers [1] [4]. The results are now briefly summarised to facilitate the discussion on the influence of gaseous impurities on hysteresis phenomena. It would seem that there is no reasonable interpretation of this influence. An attempt is made to solve the problem by considering the reasons of limited mutual solubility of compressed gaseous mixtures.

Limited miscibility of compressed gases

This type of phase equilibrium was predicted many years ago by Van der Waals and was first observed in 1940 by Krichevski and Bolshakov *). The limited solubility of gases was found in binary systems of ammonia with nitrogen [9], [10], [12], methane [11], and helium [13], and in the sulphur dioxide-nitrogen system [11]. No gas-gas equilibria were observed in binary systems of carbon dioxide with nitrogen, hydrogen and methane [15], of ethylene with carbon dioxide or nitrogen [16], and in the benzene-hydrogen system [17]. Tsiklis believed

*) A more detailed description of the phenomena will be found in papers [9] and [10].

that limited miscibility of gases may appear only in systems with one polar component; he found, however, in the course of further investigations that this property is characteristic even for systems formed by helium with ethylene [18], propane [19], or with carbon dioxide [13].

According to the continuity principle it seems reasonable to suppose that, at high pressures, gases would behave more or less like liquids. The symmetry of gas-gas coexistence curves indicates that the mixtures behave as regular solutions. Thus, Hildebrand's [20] solubility parameters $\Delta = (-E/V)^{1/2}$ may be used as a criterion of mutual miscibility of the components. The larger the difference in Δ of the components (and, to first approximation, the larger the sum of molar volumes V [20]), the more probable becomes the separation of their mixture into two liquid phases. According to Van Dranen's hypothesis [21], at the critical point (T^c , and $V=V^c$) the negative potential energy E is equal to $\frac{3}{2}RT^c$. Then $(\Delta)^2 = \frac{3}{2}RT^c/V^c$. These values for the components of the systems under consideration are given in Table I. In Fig. 1 the critical tempe-

TABLE I

Values of the squares of solubility parameters at the critical point

	T^c °K	V^c litre/mole	$(\Delta)^2$ atm.
He	5.23	0.0577	11.2
H ₂	33.25	0.0650	63.0
N ₂	126.0	0.0902	172.0
C ₂ H ₆	370.0	0.200	227.7
CH ₄	190.7	0.099	237.1
C ₆ H ₆	562.6	0.260	266.4
C ₂ H ₄	283.1	0.124	281.0
CO ₂	304.2	0.0939	398.8
SO ₂	430.4	0.1237	428.3
NH ₃	406.1	0.0729	685.5

perature-composition curves corresponding to liquid-vapour (broken lines) and gas-gas (full lines) are presented. The value of $(\Delta)^2$ for NH₃ is about 3 to 4 times higher than that of CH₄ and N₂. The limited miscibility of gases in these systems starts below T^c of NH₃ at pressures equal to about 1,500 atm. In the SO₂-N₂ system (smaller difference in Δ) a pressure of about 4,500 atm. was necessary to obtain the appearance of two gas layers at 38°C. The critical curve of this system was not drawn because only two gas-gas critical points are available. In the CO₂-N₂ system no gas-gas equilibria were observed at 0° under pressures ranging up to 7,000 atm.: the difference of $(\Delta)^2$, and the sum of V^c of the components, is still smaller than in the SO₂-N₂ system. Helium has a very low

value of $(A)^2$ in relation to other substances, and therefore the limited miscibility of gases begins at the critical point of the second component, i. e., at a relatively high temperature and under low pressure (below 200 atm.).

It is interesting that, in spite of the very low value of $(A)^2$, hydrogen does not show any gas-gas equilibria with C_6H_6 and CO_2 . This may be explained by considering the curves in Fig. 1. All the curves approach a constant value of critical composition x_1 . This limiting value is, for systems containing helium, much smaller than for any other. This is due to the large compressibility of He (because of its low internal pressure) and therefore, at high pressures, there is no space available for the second component in the helium-rich phase. On the other hand, small helium molecules penetrate easily into the interior of the second phase. The latter property shows hydrogen to a still greater extent than helium, and therefore the gas phases of H_2 with other substances (except He), if they really exist, would be very rich in H_2 ($x_1 \rightarrow 0$). The limited miscibility of He and H_2 can be predicted owing to the small differences of their molecular diameters and to the $(A)^2$ -value of H_2 , which is six times higher than that of He. It was observed to be below T^c of He [22].

In general, the criteria of solubility mentioned make it possible to predict qualitatively the appearance of gas-gas equilibrium.

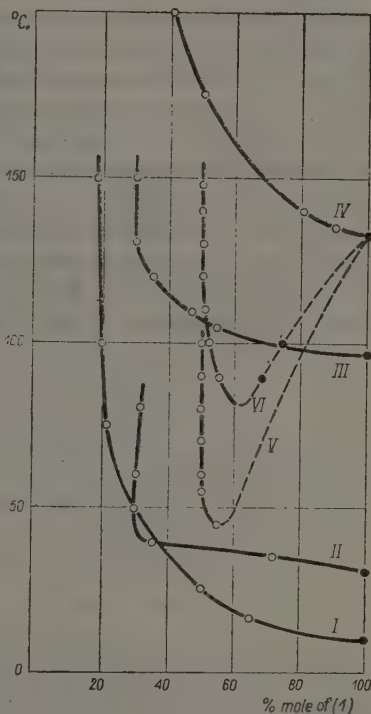


Fig. 1. Liquid-vapour and gas-gas critical temperature curves of the systems: C_2H_4 -He (I), CO_2 -He (II), C_3H_8 -He (III), NH_3 -He (IV), NH_3 - CH_4 (V), and NH_3 - N_2 (VI). Symbol (1) represents the less volatile component of a given system. The broken lines and full circles correspond to liquid-vapour critical points. References are given in the text

Hysteresis phenomena in the critical region of liquids

According to Traube, at and above the temperature of disappearance of the meniscus T_m , the coexisting phases become mutually soluble at all ratios. This statement is true only in regard to pure substances, sealed in very short tubes. For example, the changes of specific heat c_p with temperature [5] and light scattering measurements [6] indicate in fact that the system above T_m may be considered as a gaseous solution

of clusters of different sizes. Świątosławski suggested the term *gaseosol* as characterising this state of matter [7]. If the gaseosol is enclosed in a longer tube (e. g., 15 to 20 cm.), clusters tend to settle on its bottom, so that the system becomes heterogeneous, although there is no distinct interface. Fig. 2 represents schematically the vertical changes of density ρ of a fluid in a sealed tube of length h (a) just below, and (b) just above T_m (the mean density is equal to the critical one). In connection with difficulties in attaining equilibrium states in the critical region, we believe that methods based on determining the changes of density (or of a related quantity) with time after mixing, are preferable [4], [6]. Changes of the equilibrium value of $\Delta\rho$ with temperature in a heated gaseosol (A), starting at temperatures lower than T_m , and in a cooled gaseosol (B), starting at $T \gg T_m$, are plotted in Fig. 3*). The system in state B is homogeneous until T_m is reached. As the meniscus reappears,

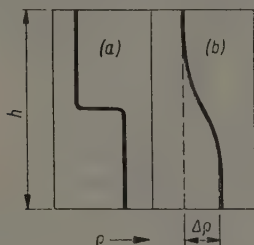


Fig. 2.

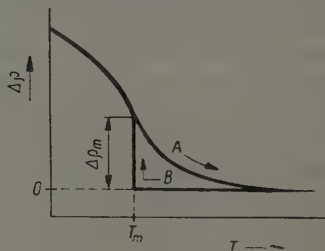


Fig. 3.

the value of $\Delta\rho$ rises rapidly to the initial value $\Delta\rho_m$. When plotting c_v [5], compressibility coefficient β , and pressure P [3], against temperature, similar hysteresis loops were observed. The differences between state A and B at a given temperature are characterised by the inequalities:

$$(1) \quad (\Delta\rho)_A > (\Delta\rho)_B; \quad (2) \quad (c_v)_A > (c_v)_B; \quad (3) \quad P_A < P_B; \quad \text{and} \quad (4) \quad \beta_A > \beta_B.$$

At the transition from state A to B, the size of clusters is reduced (ineq. (2) and (3)). In addition, the gravitational effect (proportional to β) decreases, and therefore the system in state B is homogeneous [3].

We will consider now the changes of the hysteresis loop of density upon addition of small amounts of a second component. If a non-volatile substance, possessing a high molecular weight, is added, clusters of the main component will concentrate around the surface of the large molecules. The value of $\Delta\rho$ should then increase, both in state A and B. This was confirmed in all investigations. If, however, the molecules of the

*) For reasons given elsewhere [4], the scheme shown in Fig. 3 differs somewhat from that obtained by McIntosh and Maass.

substance being added are small and their interaction energy is relatively low, i. e., the substance is gaseous at the critical point of that being in excess, there is no reason for an increase in size of the existing clusters. In addition, the value of β , and even the gravitational effect doubtless decreases, and thus the mixture should become homogeneous just above T_m . Contrary to this conclusion, McIntosh, Dacey and Maass [8] have observed that, after addition of traces of air to ethylene, the heterogeneity of the gaseosol in state *A* increased and *reappeared* on cooling, reaching almost the same value at the given temperatures as in state *A*. The possibility of explaining this phenomenon as a certain kind of limited miscibility of gases is doubtful owing to the low density and pressure of the system and the small differences in $(\Delta)^2$ -values of N_2 and C_2H_4 (no gas-gas equilibria were found in this system at 0°C. up to 7000 atm.).

Tsiklis [18] observed visually and has photographed the gas-gas interface in the C_2H_4 -He system at 16° (250 atm.). It is of the type 2(a), corresponding to a distinct meniscus. It is, however, not quite certain whether such distinct interface exists when the differences in $(\Delta)^2$ of the components are less than in the C_2H_4 -He system, or when the temperature of a given system increases. It seems that with decreasing difference in solubility parameters of the components, the gas-gas interface may change gradually from type (a) to type (b) (Fig. 2), corresponding to a state of incipient phase separation.

Summary

The limited mutual solubility of compressed gases may be qualitatively predicted if the solubility parameters Δ of the components are known. Δ -values have been calculated at the liquid-vapour critical points on the basis of van Dranen's hypothesis. The influence of gaseous impurities on hysteresis phenomena is discussed. Phenomena due to this influence seem to be a certain kind of limited miscibility of two gases.

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The Electric Conductivity and Catalytic Activity of MgO-CdO Mixed Catalysts

by

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Presented by B. KAMIENSKI on April 4, 1957

It was shown in our previous investigations [1]–[4] that in the case of catalytic ethyl alcohol dehydrogenation there exists a simple relation between the change of the electric conductivity of the catalyst and the yield of catalytic reaction:

$$(1) \quad r = A + B \cdot |\Delta \log \sigma|,$$

where r — catalytic reaction yield; $|\Delta \log \sigma| = \log \sigma_2 - \log \sigma_1$; σ_1 and σ_2 — the electric conductivity of the catalyst before the reaction and after it has reached a new constant level in the course of catalytic reaction; A and B — constants characteristic for a given catalyst. This linear relation was observed for n - and p -semiconducting oxides as well as for their n - n , p - p and n - p mixtures. This fact may be explained [4], [5] by assuming that the reaction yield as well as the increase of the number of free electrons (or the decrease of the number of positive holes) are functions of the number of chemisorbed molecules. It seems also that the constants A and B are influenced by the initial electronic state of the catalyst before the catalytic reaction took place. The initial conductivity σ of our catalysts which may be taken as the measure of their initial electronic state changed with temperature according to the well known exponential law. This variable however can be eliminated by using as catalyst a substance the conductivity of which is independent of temperature. The aim of the present investigation was just to carry out a series of measurements of catalytic and electric behaviour of the catalyst showing temperature-independent electric conductivity.

According to Hogarth [6] the conductivity of cadmium oxide is practically independent of temperature within low temperature range up to 200°. Cadmium oxide used by us showed also a conductivity independent of temperature within the temperature range 20–150°. This

temperature range however is too low for catalytic ethyl alcohol dehydrogenation, its velocity under 150–200° being very slow. By mixing CdO with MgO we could get a catalyst the electric conductivity of which was practically constant within the temperature range 100° up to 500°.

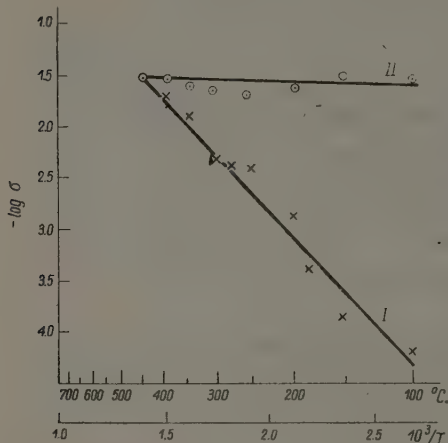


Fig. 1. Logarithm of the conductivity of CdO + 1.7 MgO mixtures as the function of reciprocal absolute temperature

After preliminary measurements of the conductivity of mixtures of various composition, the composition 1 mole CdO + 1.7 mole MgO was chosen for further investigation. On heating such mixture in air from room temperature up to 500° its conductivity increased as it is shown by curve I in Fig. 1. On cooling, however, the conductivity changes only very slightly and may be taken as practically constant (curve II, Fig. 1). It does not change any more on reiterated heating and cooling.

The measurements of the electric conductivity in the course of the catalytic ethyl alcohol dehydrogenation were

carried out in the same way as in previous investigations and with the help of the same apparatus [7]. Analytically pure MgO and CdO were used to prepare the catalyst. The powders were passed by a 200-mesh sieve, mixed and pressed with a hand press into cylindrical pellets 6 mm. high and 12 mm. in diameter. Before the reaction the catalyst was heated up to 500° and then cooled down to the intended reaction temperature.

After the vapours of ethyl alcohol-water mixture were introduced into the reaction chamber, the electric conductivity of the catalyst increased and after 5–10 minutes reached a new constant value. Fig. 2 shows the changes of $\log \sigma$ as the function of time, the moment of introduction of reacting vapours being taken as zero. Curve I shows the results of the measurements carried out at 225° with 40% (vol.) alcohol-water mixture. Curve II corresponds to 250° and 20% alcohol-water mixture. Similar curves were obtained within the temperature range 200–300° also with the mixtures of other compositions. After the current of reacting vapours was cut off and initial conditions restored the electric conductivity of the catalyst decreased always to its initial value. Curve III shows the change of $\log \sigma$ when only water vapour was introduced at 300° into the reaction chamber. It is seen that the influence of water vapour on the electronic state of the catalyst is very slight.

The $|\Delta \log \sigma|$ -value (full line) and catalytic reaction yield r (dotted line) for 20% (vol.) alcohol-water mixture are given in Fig. 3 as the function of temperature. Both values are practically constant within the temperature range 200–300°.

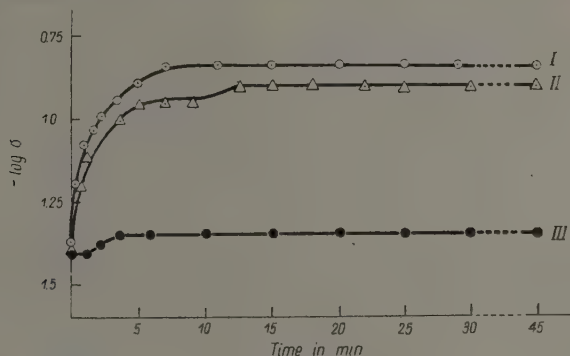


Fig. 2. Change in $\log \sigma$ of the CdO-MgO mixed catalyst in the course of the reaction of dehydrogenation of ethyl alcohol vapours. Curve I — temperature 225°, 40% (vol.) alcohol-water mixture; curve II — temperature 250°, 20% (vol.) alcohol-water mixture; curve III — temperature 300°, pure water vapour

At 450°, reduction of the CdO contained in the catalyst took place. It was followed by evaporation of metallic cadmium which condensed

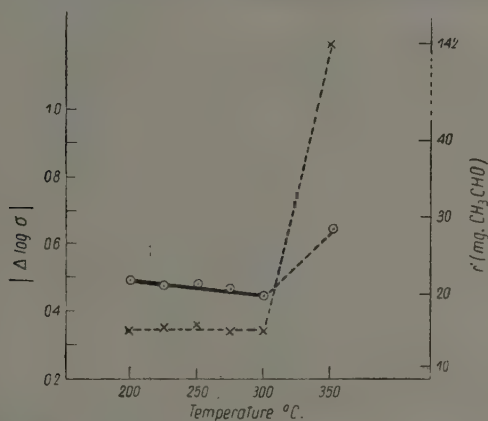


Fig. 3. Changes in the $|\Delta \log \sigma|$ -value (full line) and the catalytic reaction yield (dotted line), both as the function of the reaction temperature

on the walls of the cold parts of the apparatus. The surface of the catalyst was covered with a layer of pure MgO about 1 mm. thick and simultaneously its conductivity decreased as shown in Fig. 4. These processes were also accompanied by a strong decrease of the catalytic activity of the catalyst.

An isothermic series of measurements at 225° was also carried out with alcohol-water mixtures of various composition. The results are

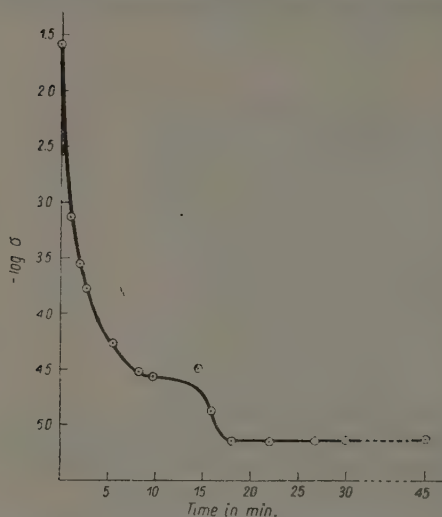


Fig. 4. Change of $\log \sigma$ of the catalyst at 450° in an atmosphere of 20% alcohol-water vapours

of the electric conductivity in reactant vapours medium. The catalyst therefore behaved as *n*-semiconductor. While heated in air the primary CdO-MgO mixture suffered some irreversible change resulting in the formation of the catalyst, the conductivity of which was independent of temperature within the wide range $100\text{--}500^\circ$. It is very probable

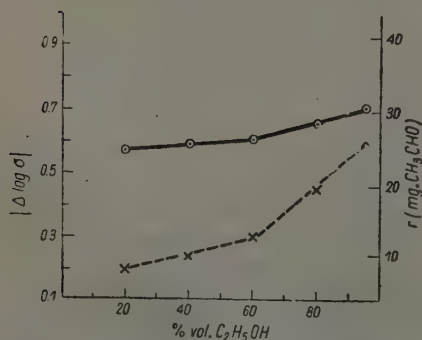


Fig. 5. $|\Delta \log \sigma|$ -value (full line) and catalytic reaction yield (dotted line) as functions of the composition of reacting alcohol-water mixture

given in Fig. 5. The full line shows the $|\Delta \log \sigma|$ -value and the dotted line the *r*-value, both as functions of the composition of reacting vapours. Fig. 6, where $|\Delta \log \sigma|$ is plotted against catalytic reaction yield *r*, shows the linear dependence of both values.

It was found in the course of previous investigations that the adsorption of the reactants in the course of ethyl alcohol dehydrogenation was accompanied by the formation of donor centres on the surface of the catalyst. In the case of the catalyst composed of 1 mole of *n*-semiconducting CdO with 1.7 mole of *p*-semiconducting MgO, we observed an increase

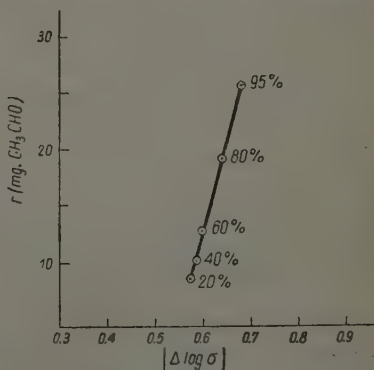


Fig. 6. The $|\Delta \log \sigma|$ -values as the function of catalytic reaction yield for alcohol-water mixtures of various composition

that, in the course of heating, surface migration of CdO (which is known as relatively volatile oxide) takes place, and the surface of MgO grains is covered with a thin layer of pure CdO or of solid CdO-MgO solution determining the electric conductivity and catalytic activity of the catalyst.

Fig. 3 shows that the $|\Delta \log \sigma|$ -value, as well as the catalytic reaction yield, are practically independent of temperature within the range 200 – 300°. Also, the initial conductivity of the catalyst (before the catalytic reaction) is constant within this temperature range. These facts are in agreement with the views that the number of current carriers of the catalyst before the reaction influences sorptive equilibria in the course of the catalytic reaction and, by their intermediation, influences also catalytic activity and $|\Delta \log \sigma|$ -value. When the number of current carriers is kept constant, although the temperature is not, the same sorptive equilibria are reached and we get the same reaction yield and practically the same $|\Delta \log \sigma|$ -value. Of course, this is only possible if the energy of activation of the catalytic reaction is low.

At temperatures above 300° we observed an increase of the $|\Delta \log \sigma|$ -value as well as an increase of the catalytic reaction yield. The authors think that these changes are due to the slow reduction of CdO which is fast at 450°. At 350° the reduction may result in the formation of a number of defects in the surface layer of the CdO lattice thus increasing its catalytic activity, initial conductivity and the $|\Delta \log \sigma|$ -value.

The decrease of all three values at 450° is evidently due to the evaporation of metallic cadmium from the surface of the catalyst grains. The resulting MgO layer is catalytically inactive and shows high electric resistance [8]. The curve in Fig. 4, showing a rapid decrease in conductivity of the catalyst, illustrates, therefore, the process of formation and thickening of such an MgO layer.

The results of our experiments with alcohol-water vapours of various composition show that linear relation (1) does hold not only when the substrate of constant composition reacts at various temperatures but also when mixtures of various compositions react at constant temperature.

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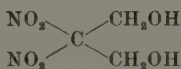
On Ultraviolet and Infra-Red Absorption Spectra of 2,2-Dinitropropane-1,3-diol

by

Z. ECKSTEIN, T. URBANŃSKI and W. SOBÓTKA

Presented by T. URBANŃSKI on April 10, 1957

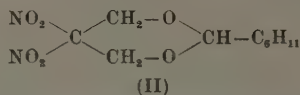
Continuing our former experiments on nitroalcohol [1]–[5] we next examined the spectra of 2,2-dinitro-1,3-propanediol (I). This compound can be easily prepared by reacting dinitromethane (in the form of its potassium salt) with formaldehyde, according to the methods described in the literature [6], [7].



(I)

According to Feuer, Bachman and May [8] the compound possesses interesting properties: it reacted with amines forming polinitropiperidine derivatives, but did not form derivatives of tetrahydrooxazine or hexahydropyrimidine, as might be expected from the former work on the reaction of primary nitroparaffins or nitro-1,3-diols with formaldehyde and amines [9]–[11]. Compound (I) differs also from the other nitro-1,3-diols by the fact that it colours skin-proteins orange. The colour changes into bright yellow under the action of alkali.

The ultraviolet absorption spectrum of (I) shows an interesting anomaly of two maxima (Fig. 1, Curve 1). There is one weak maximum (a) near $276 \text{ m}\mu$ ($\Delta\epsilon=4$), typical for the nitro group. It is much weaker than in nitroalcohols (where $\Delta\epsilon=9.4\text{--}13$) but stronger than in nitro-1,3-diols ($\Delta\epsilon=0\text{--}2$). Another maximum (b) near $363 \text{ m}\mu$ we suggest attributing to the existence of two NO_2 chromophors influenced by two alcoholic groups. To check this hypothesis, cyclic acetal, i. e., 5,5-dinitro-2-cyclohexyl-1,3-dioxane (II) was prepared by the action of cyclohexylformal-



dehyde on the diol (I), and the absorption spectrum of the new 1,3-dioxane derivative was examined. The absorption curve (II) is shown in Fig. 2. It contains only one very weak maximum ($A\epsilon=2.5$) near to

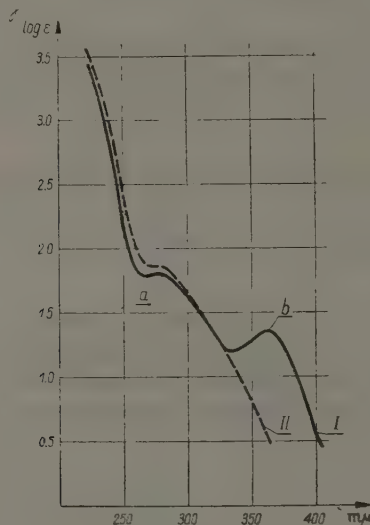


Fig. 1

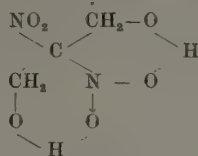
278–280 $m\mu$. It is interesting to note that 2,2-dinitropropane shows a maximum at the same wave-length, i. e., 280 $m\mu$ [12]. The absence of the second maximum suggests that the presence of two hydroxyl groups can be responsible for the existence of the second absorption maximum of compound (I).

TABLE I

Characteristics of ultraviolet absorption spectra of compounds (I) and (II) in ethanolic solution

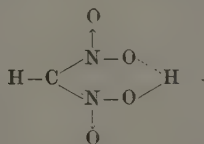
Compound	Max.		Min.		Max.		Min.	
	λ $m\mu$	ϵ	λ $m\mu$	ϵ	λ $m\mu$	ϵ	λ $m\mu$	ϵ
I	276	64	267	60	360	21	334	15
II	279	72.6	270	70.1	—	—	—	—

Structure (Ia) could indeed be admitted on the line of our former hypothesis [1], [2]. One nitro group is bonded with two hydrogen bonds and thus the second nitro group is "completely free" and would give the second maximum at $360\text{ m}\mu$.



(Ia)

The strong bathochromic shifting of the maximum may be attributed to the effect of two chelate rings, which could possibly be considered as auxochroms. Also an earlier work by Hantsch and Voigt [13] shows that the potassium salt of *aci*-dinitromethane gives a deep maximum near $306\text{ m}\mu$. The bathochromic shifting of the maximum is produced, according to Hantzsch and Voigt, by the formation of a new cyclic chromophor (III) with a hydrogen bond between one *aci* and one normal nitro group.



(III)

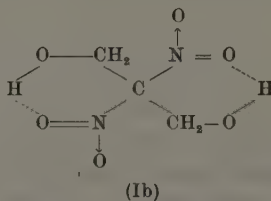
Diagram (III) is obviously not applicable in our case, as compound (I) cannot of course produce the *aci* form. To check the hypothesis on the structure (Ia) of 2,2-dinitropropane-1,3-diol, the infrared absorption spectrum of the compound was examined.

Compound (I) was used either in the form of mull in paraffin oil, the ratio of substance to paraffin oil being 1:8, or as diluted solution in carbon tetrachloride. The spectra were measured with a Hilger single-beam spectrophotometer, using rock-salt optics. The results are tabulated (Table II) and shown in the diagram (Fig. 2).

The following are the main features of the results thus obtained. No band typical for the free hydroxyl group ($3590\text{--}3650\text{ cm}^{-1}$) can here be seen. Only a relatively weak maximum at 3174 cm^{-1} was observed in carbon tetrachloride solution. This would indicate that both hydroxyl groups are bound through the internal hydrogen bonds. When solid substance was examined, no maximum but only a bend was noticed at the same wavelength. This may be due to the association through the intermolecular hydrogen bond.

The intra- and intermolecular hydrogen bonds are probably formed through the nitro group, and the former could most likely be expressed

by the structure (Ia). This structure is supported by the three maxima characteristic for the nitro group: 1587, 1568, 1544 cm^{-1} . According to the previous work on infra-red spectra of nitroparaffins and their hydroxyl derivatives [4] the frequency of NO_2 -bands is reduced by the hydrogen bonds (1544 cm^{-1}). The intermediate frequency 1568 cm^{-1} could be produced by the "partially bonded" nitro group according to the diagram (Ib):



The frequency 1587 cm^{-1} should be assigned to the "free" NO_2 group.

The same three maxima, shifted towards lower frequencies, can be found in the spectrum of (I) dissolved in carbon tetrachloride: 1568,

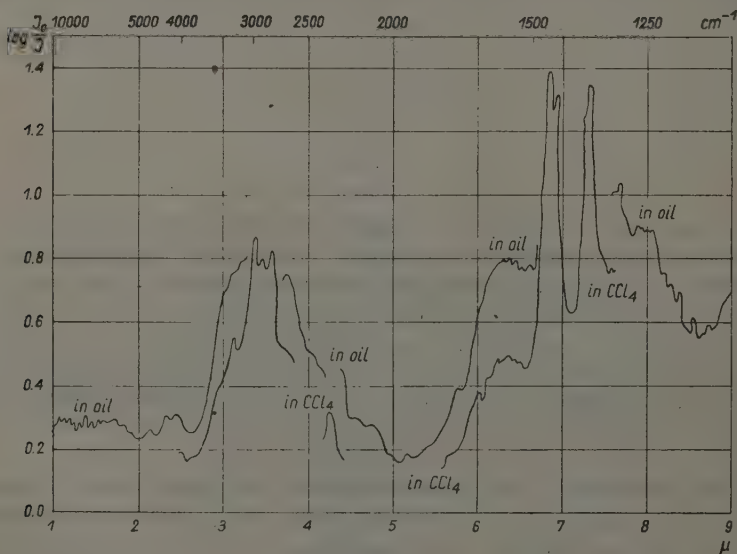


Fig. 2

1562 and 1527 cm^{-1} . As mentioned above we suggest assigning these frequencies to free, "partially bonded" and bonded nitro groups, respectively.

In the region of symmetric stretching NO_2 vibrations, two bands were registered when substance (I) was examined in solution in carbon

TABLE II
Characteristics of infra-red absorption bands

	Frequency (cm. ⁻¹)	$\log \frac{I^0}{I}$	Assignement
Paraffin oil mull, thick- ness of layer 0.1 mm.	9090	0.2954	
	8510	0.2858	
	8161	0.2907	
	7843	0.2806	
	7547	0.2814	
	7142	0.2907	
	6779	0.2806	
	6451	0.2882	
	5714	0.2934	
	4651	0.2603	
	4255	0.2985	
	4000	0.3054	
Solution in CCl ₄ , thick- ness of layer 0.02 mm. concentr. c. 0.002 M/l.	3174	0.5406	
	2985	0.8631	
	2898	0.7986	
	2816	0.8240	
	2350	0.3205	
Paraffin oil mull, thick- ness of layer 0.1 mm.	2197	0.2865	Asymmetric stretching NO ₂ vibrations forming a wide band with multiple maxima
	2105	0.2700	
	1941	0.1721	
	1724	0.3959	
	1587 free	0.7932	
	1568	0.8013	
	1544 bonded	0.7905	
	1487	0.7746	
Solution in CCl ₄ thick- ness of layer 0.02 mm. concentr. c. 0.002 M/l.	1652	0.3726	Asymmetric stretching NO ₂ vibration forming a wide band with multiple maxima
	1632	0.4294	
	1600	0.4863	
	1568 free	0.4976	
	1562	0.4976	
	1527 bonded	0.4609	Symmetric stretching NO ₂ vibrations
	1459	1.3872	
	1440	1.3098	
	1370 free	1.3468	
	1324 bonded	0.7773	
Paraffin oil mull, thick- ness of layer 0.1 mm.	1310	1.0178	
	1265	0.9101	
	1242	0.8997	
	1215	0.7619	
	1208	0.7281	
	1190	0.7100	
	1169	0.6234	

tetrachloride. They were: 1370 cm^{-1} (free) and 1324 cm^{-1} (bonded NO_2 group).

The cyclic structures (Ia) and (Ib) seem also to be supported by the CH_2 vibrations 1440 and 1459 cm^{-1} . Their frequencies correspond to CH_2 in cyclic compounds (1448 – 1456 cm^{-1}) and in the neighbourhood of the double bond [14], [15], which in our case is the $-\text{N}=\text{O}$ bond.

It is worth mentioning that according to Feuer, Bachman and May [8] 2,2-dinitropropane-1,3-diol does not show the presence of hydroxyl groups (reaction with cerium nitrate [8]).

Further experiments on 2,2-dinitropropane-1,3-diol and experimental details will be reported elsewhere.

Our thanks are due to Miss D. Ciecierska and Miss K. Gorczyńska for examination of ultraviolet absorption spectra; to Miss T. Kraczkiewicz and Miss B. Kontnik for examination of infra-red absorption spectra; and to J. Świątosławska, D. Sc. under whose direction the spectral analysis was carried out.

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Cross-Section of the Structure of Cieszyn

by

K. KONIOR and A. TOKARSKI

Presented by J. SAMSONOWICZ on April 19, 1957

Three deep borings in the vicinity of Cieszyn — P1, C1 and C2 — have revealed the peculiar structure of the metamorphic and Palaeozoic substratum of the *in situ* overlying Miocene beds, and higher up that of the overthrust Carpathian Flysch. In A. Tokarski's earlier paper [1] on the structure of Cieszyn borings, C1 and C2 were lettered X and Y.

The profiles of the boreholes were based on cores, character of bore-mud and electric logging. The synthetic profile is as follows (bracketed figures refer to section in Fig. 1a):

Overthrust Carpathian Flysch; the Cieszyn unit, 484 to 645 m. thick, consists of the subunit of Goleszów (Ia) and that of Wiślica (Ib); we note here upper Cieszyn shales — black, marly, with sandstone intercalations (18); Cieszyn limestones (17), lower Cieszyn shales with limestone intercalations (16) and a teschenite sill (*t*). These formations of Berriasian-Valanginian age retain their lithological character within the whole of the Carpathian arc area. The sub-Silesian unit (II) with the Eocene and Upper Cretaceous deposits, from 55 to 220 m. thick, consists of green and red shales — partly marly — and of black shales (15).

Below it lies the Helvetian, pierced in borehole C2 (14), where it is 33 m. thick, made up of grey argillaceous shales with conglomerates; *Vaginella depressa* was recorded here by W. Krach.

Miocene, in boreholes C1 and C2, forms a scale (13a) consisting, on its base, of green clays with quartz pebbles and a foraminiferal fauna of the Miocene (*Nonion boueanum*), of the Eocene (*Recurroides walteri*) and of the Upper Cretaceous (*Globotruncana arca*); the last two forms were washed out from Flysch deposits. Here lie *in situ* Dębowiec sandstones and conglomerates (12) containing pebbles of quartz and those of metamorphic and Carboniferous rocks, their abundance gradually increasing southward. Finally, there are overlying grey argillaceous shales with a fauna of Toronian *Globigerina*, *Candorbulina* and *Globorotalia* (13).

Substratum, Carboniferous: Sandstones with coal veins, after T. Bocheński, of the Saddle beds (11); underlying these are Namurian Border

beds (10), 600 m. thick; they are represented by shales from which J. Gozdysz has recorded *Rhodea tenuis* and *Sphenopteris schlehani*; they are intercalated by sandstones with kaolin matrix (according to the authors, displaying a "molasse" facies). In these beds, boreholes P1 and C1 showed up a sill of rock of the diabase type (d), one metre thick in borehole P1 and growing thicker to the North.

The underlying "passage" series, 170 m. thick, consists of shelly Flysch-like sandstones containing mica, with occasional intercalations of black shales and "molasse" sandstones (9). They carry plant detritus only. Further down the authors' "Flysch" series (8), 320 m. thick, is made up of dark shales; after K. Korejwo its higher part, 280 m. thick, contains a Lower Namurian fauna such as *Ctedononta transversalis*, *Bucaniopsis moravicus*, *Eumorphoceras*, *Nuculoceras* and others. Towards the top, this fauna grows poorer and consists of more distinctly euryhalic forms. Here, after K. Korejwo, lies the boundary line between the Namurian and the Visean, since the lower beds (7) contain layers of oölitic and *Syringopora* limestones as well as limestone intercalations yielding a copious assemblage of Visean fauna with *Gigantoproductus*. The Lower Visean strata are made up of sandstones, 15.45 m. thick, with, at the top, two thin tuffite layers.

The Devonian, ca. 670 m. thick, consists solely of carbonaceous rocks and has been divided by J. Samsonowicz into four stratigraphic stages:

The Famennian (6) up to 180 m. thick — dark limestones with a dolomite intercalation, unfossiliferous except for an indeterminate trilobite (from the Proetidae family?).

The Frasnian (5), ca. 120 m. thick, limestones with *Amphipora pervesiculata* Lecompte.

The Givetian (4), up to 200 m. thick, has been ascertained on the presence of numerous *Spirifer mediotextus* Arch. & Vern. and *Amphipora ramosa* Phill.

The Eifelian? (3), limestones and dolomites with indeterminate brachiopods, *Spirifer* sp. and others.

Metamorphic. These are biotite schists (2), after W. Żabiński garnet-bearing, dipping 45°. The underlying gneisses were pierced over a depth of 1.8 m. (1). According to W. Żabiński we have here two-mica gneisses with veins of quartz.

The quartz veins encountered within the gneisses may be responsible for the hitherto puzzling quartz pebbles occurring in the post-Hercynian sediments of south-western Poland. The biotite schists of the substratum may have been the source which provided the material for the "biotite Cretaceous" of the units of the Magura overthrust [2].

Veins of calcite, and silica and pyrite impregnations may, in L. Kozarski's opinion, indicate the proximity of some intrusions, similarly as is the case within the Devonian of Dębnik [3]. The Devonian (probably Eifelian) in the profile of borehole P1 lies transgressively on the Metamorphic,

while the Upper Viséan transgresses on the Famennian, as on the western margin of the Upper Silesian Basin, in Moravia [4], [5].

The Flysch facies of the Culm in Puńców (P_1) continues high up into the Namurian, contrary to the western marginal area of the Upper Silesian Basin, where it terminates sharply on the boundary between the Viséan and the Namurian. This fact is most probably connected with

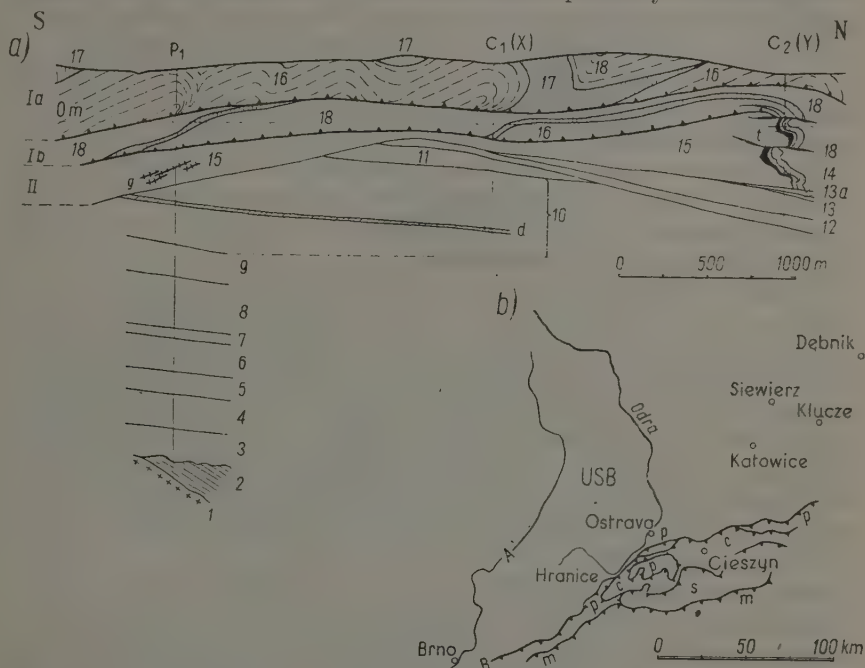


Fig. 1 a) Cross-section of the Structure of Cieszyn. Explanation in the text
 b) Sketch map. Explanation: A—border of Sudeten Mts., B—border of Carpathian Mts., USB—Upper Silesian Carboniferous Basin. Overthrust units of Carpathian Flysch: p —sub-Silesian Unit, c —Cieszyn Unit, s —Silesian Unit, m —Magura Unit

the longer duration of the sea in the south, where some as yet unknown connection with the Tethys may have existed.

The Palaeozoic, within the substratum of Miocene and Flysch deposits, forms a ridge with a denivellation of not less than 400 m. Upper Carboniferous rocks on this ridge show a reddish colouration due to weathering, over a thickness of 26 m. from the top downwards.

The Miocene in the Flysch substratum, lying *in situ* (12, 13), occupies the northern ridge slope, and thickening to the north it attains in borehole C_2 a thickness of up to 160 m.; it displays no signs of tectonic engagement. The absence of Miocene deposits in the southern slope of the Palaeozoic ridge in P_1 is, in the authors' opinion, due to its removal by the Flysch masses shifting to the north.

The Helvetian in borehole C2 (14) constitutes a scale at the bottom of the sub-Silesian unit which has been dragged here from the south together with Flysch reliefs in its substratum (13a).

The Flysch masses were overthrust far to the north, covering the molasse beds which had sedimented during the Miocene; from the south the Helvetian (14) and the reworked Flysch scale (13a) had been brought by them, as were probably also gypsum fragments ascertained within the Flysch of the sub-Silesian unit in borehole P1 (Fig. 1a. g).

Deposits of the sub-Silesian unit (II, 15) thin out on culmination of the substratum ridge smoothing out its relief; they served as tectonic lubricant for the masses of the higher overthrust units.

The Wiślica sub-unit (Ib), with monoclinal dips between P1 and C1, in borehole P1 reveals scale-doubling due to pressure from the south. The upper Golezów sub-unit (1a) consists of four crowded folds, as has been established by M. Książkiewicz [6].

Four, cascade-like, recumbent folds, dissected by two horizontal dislocations, are formed to the north of the ridge by the sub-Silesian (II) unit jointly with the Wiślica sub-unit. Consequently, along the line of borehole C2, there is an additional swelling of the Flysch masses which the Golezów sub-unit (1a) continues to override. Thus, the palaeomorphology of the older substratum has but little bearing on the modelling of surface tectonics. This is all the more amazing in view of the fact that the older substratum may even pierce to the surface through the overthrust Flysch, as has been shown by Homola [7] on the example of Hranice in Czechoslovakia.

Kind acknowledgements are due to all those who have co-operated in the description of results of drilling work done in the vicinity of Cieszyn, namely, in addition to the persons mentioned in the text, also to I. Heller, E. Huss, S. Liszka, W. Parachoniak and Professor K. Zapletal of Brno.

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On the Non-Marine Lamellibranchs from the Namurian of the Bug Basin

by

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Presented by J. SAMSONOWICZ on April 19, 1957

The existence of the Bug Basin has, in 1939 [1], been ascertained by boreholes. Following the Second World War extensive investigation work undertaken by Soviet geologists [2] has revealed the structure of this Basin sufficiently to enable the economic working of coalfields in the vicinity of Sokal on the Bug River.

In 1942, a borehole was sunk at Strzyżów, north of Sokal, to a depth of 830 m. Below Cretaceous strata, at a depth of 394.6 m., the borehole attained Carboniferous deposits. A description of the Strzyżów Carboniferous has been published by M. Schwarzbach in a short note [3] and later in a more comprehensive paper [4]. His inferences, though mainly correct, nevertheless call for some amendments and additions, as has been ascertained by the writer during her work of describing samples yielded by that borehole.

The synthetic profile of the Strzyżów Carboniferous, determined on abundant fauna, is as follows:

- 394.6 to 490.7 m. Middle Namurian deposits — argillaceous shales with a limestone intercalation at the bottom and several sandstone intercalations; at the top they correspond to the Poręba strata of the marginal (Ostrawa) series in the Rybnik district of the Upper Silesian Basin.
- 490.7 to 629.0 m. Lower Namurian — argillaceous shales with scarce limestone and sandstone intercalations; an equivalent of the Lower Ostrawa strata in the Upper Silesian Basin.
- 629.0 to 716.9 m. Upper Viséan — shales and limestones.
- 716.9 to 755.0 m. Middle Viséan — limestones and shales, the latter predominant.

755.0 to 830.0 m. Calcareous conglomerates and mudstones, detritic — lower Middle Viséan or higher Lower Viséan (not pierced).

Within the higher zone of the Strzyżów Namurian the writer has ascertained the occurrence of a series of deposits, nearly 20 m. thick, described by Schwarzbach in a somewhat cursory manner, which, how-

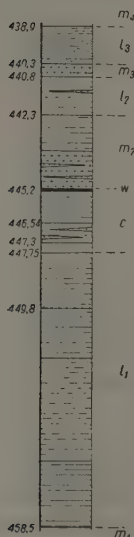


Fig. 1. Profile of a series with lagoonal sediments in the Namurian of Strzyżów. Explanation in the text.

ever, merits special attention owing to its facial and faunal character. The profile of this series is shown in Fig. 1. It consists of lagoonal deposits (l) separated by continental sediments (c), with, at top, two marine zones (m). Overlying the marine deposits (m_1) is a formation, nearly 11 m. thick (between 447.75 to 458.5 m.), of argillaceous shales and claystones (l_1); they are non-calcareous, with the exception of one thin calciferous intercalation at 449.8 m. Predominant here are non-marine Lamellibranchs of genera *Anthraconaia* and *Anthraconauta* whose state of preservation indicates environment common in their usual habitat, i. e., a fresh-water lagoon. The inference drawn as to the lagoonal character of the deposits is not cancelled by the occurrence there of *Spirorbis* sp., *Serpula* sp., *Lingula mytiloides* Sow. (a whole layer) and *Modiola* cf. *meeki* de Kon., as euryhalic forms, or that of fossil fishes. The sporadic occurrence at a depth of 449.8 to 450.1 m. of cephalopods, *Pseudorthoceras* cf. *striolatum* (v. Meyer), *Homoceras* cf. *smithi* (Brown) and *Reticuloceras* sp., is in that environment probably due to strong tides which brought them into the lagoon.

Higher up, at a depth between 445.2 and 447.75 m., are continental deposits (c), fine-grained sandstones streaked by clay and separated by a bed of claystone with arenaceous intercalations, carrying abundant mica and plant detritus, at the top containing *Stigmara*. The upper part of these beds is a typical rootled-bed covered by a thin vein of coal (w).

The beds between 442.3 to 445.2 m. indicate sea transgression (m_2). We encounter here argillaceous shales sediment, calcareous at bottom, with thin sandstone intercalations, containing the marine fauna: *Crinoida*, *Entalis* sp. (a whole layer), also *Lingula* sp. and a shell of *Anthraconauta belgica*, most likely brought here by a current.

At a depth of between 440.8 and 442.3 m. non-calcareous argillaceous shales have again been deposited in the lagoon (l_2); there is an abundance of *Anthraconaia* and *Anthraconauta*; *Lingula mytiloides* is present, as also numerous fish remains within the arenaceous intercalation.

The only specimen of a goniatite has probably been brought here by the action of waves.

Between 440.3 to 440.8 m. there is again a marine sediment (m_3): calcareous argillaceous shale, black or grey with an abundant, distinctly marine fauna: *Nucula gibbosa* Flem., *Posidonia corrugata* Eth., *Naticopsis* sp., *Orthoceras* sp., *Homoceras* cf. *smithi* (Brown), *Reticuloceras* sp.

A third lagoonal series (l_3) appears between 438.9 and 440.3 m. containing abundant *Anthraconaia* and *Anthraconauta*; a coral fragment collected here is due to the action of waves.

Within shales of the higher layers non-marine Lamellibranchs disappear; a fauna with *Myalina* and *Lingula* suggests gradually increasing salinity in the lagoon, while a copious and distinctly marine fauna makes its appearance at 436.6 m.

Deposits lying between 438.9 and 458.5 m. represent the upper part of a sedimentary cycle terminating in a coal vein at a depth of 445.2 m., and the lower part of a higher sedimentary cycle during which lagoon sediments reappear twice among sediments of marine character.

Among representatives of non-marine Lamellibranchs of the genera *Anthraconaia* and *Anthraconauta* the writer has ascertained the presence, within deposits of the described profile, of the following species: *Anthraconaia* cf. *bellula* (Bolton), *Anthraconaia lenisulcata* (Trueman), *Anthraconaia tchernyshevi* Kor., *Anthraconauta belgica* (Hind), *Anthraconauta samsonowiczi* Kor., *Anthraconauta truemani* Kor. These forms make up a fairly characteristic assemblage which the writer regards as typical for the Poręba substage within the Ostrawa strata of the Rybnik district in Upper Silesia [5].

Earlier students, such as Hind (1893), A. Schmidt (1910), Pruvost (1919), and Tchernyshev (1931) have stressed that the non-marine fauna in the Westphalian must have derived from forms which had initially lived in marine environment and had gradually migrated to the brackish and fresh waters of river estuaries. Consequently, in the Namurian, these forms may, all the more so, be supposed not to avoid brackish waters.

It is striking that representatives of the Mytilidae family from the Rybnik Namurian — *Anthraconaia*, *Anthraconauta* and *Naiadites* — are "Modiola-like" in shape. It must be added that in a number of specimens of the genera *Anthraconauta* and *Naiadites* the writer has ascertained in the prismatic stratum the presence of striae, transversal to the growth lamellae, common in the genera *Mytilus* and *Modiola*.

It is to be supposed that the genera *Anthraconauta*, *Anthraconaia* and *Naiadites* evolved from the genus *Modiola* of the Mytilidae family. Hence, some forms among the Namurian representatives of these genera are, particularly at an early stage of evolution, barely distinguishable from genus *Modiola*.

We may infer that the separation of the genera *Anthraconaia*, *Anthraconauta* and *Naiadites* may have taken place in the Dinantian; the end of this process is referable to the Namurian age.

Within the southern part of the Bug Basin, called by Soviet geologists the Galician-Volhynian depression, P. Shulga [2] ascertained in the Upper Namurian (Bug-zone), in addition to a scanty marine fauna, also the presence of non-marine forms, mainly *Anthraconauta bugensis* Shulga and *Anthraconauta belgica* (Hind), as well as some rare *Naiadites*, *Carbonicola pseudacuta* Trueman and *Carbonicola praeaquilina* Shulga. Beds yielding this fauna apparently are an equivalent of the freshwater horizon in the Strzyżów borehole.

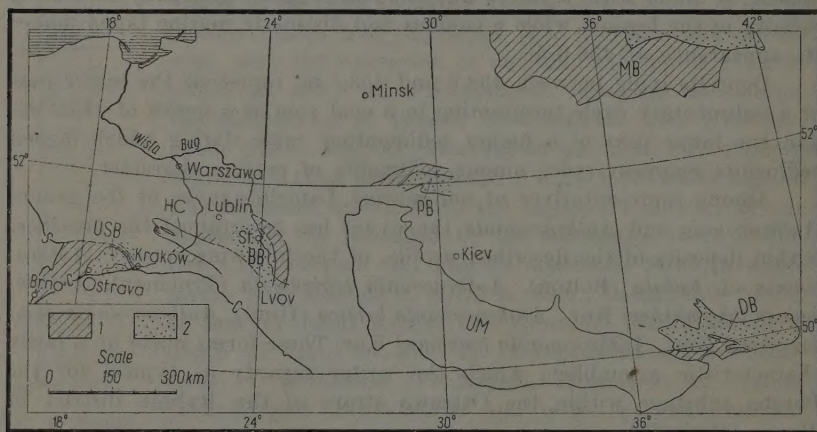


Fig. 2. Sketch Map of Carboniferous Basins in eastern Poland and the south-western part of USSR.

Explanation: *S*—border of Sudeten Mts., *O*—border of Carpathian Mts., *HC*—Holy Cross Mts., *UM*—Ukrainian Massif, *USB*—Upper Silesian Basin, *BB*—Bug Basin, *PB*—Pripyat Basin, *DB*—Donets Basin, *MB*—Moscow Basin; *St*—borehole in Strzyżów; 1—Lower Carboniferous, 2—Middle and Upper Carboniferous.

Within the Bug Basin the marine fauna contains numerous components in common with the Upper Silesian Basin; the writer believes the described fauna of non-marine Lamellibranchs from the Strzyżów Namurian to be identical with that from the Rybnik district. There must no doubt have existed between these Basins some close connection favouring mutual faunal migration. This migration, as concerns marine forms, took place by a route north of the Holy Cross Mountains (Fig. 2). The non-marine fauna may have spread along the shores of then existing marine basins, where fresh water reservoirs were present; possibly it may have been transported in the larval stage by sea currents, since it is not inadmissible that larvae were more euryhalic than adult individuals.

The connection between the Upper Silesian and Bug Basins seemed quite obvious also to Bederke [6] and Schwarzbach [4]; neither have these authors any doubt as to the connection between the Bug Basin with the Donets and Moscow Basins.

On evidence of the latest investigation we may infer that this juncture existed through the Pripyat Basin, where Dinantian sediments have been studied sufficiently to show marked similarities with the Moscow Basin; the knowledge, however, about Upper Carboniferous sediments is still inadequate [7].

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